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Chapter One

Introduction

The concept of a unipotent representation has its origins in the representation theory of finite Chevalley groups. Let $G(\mathbb{F}_q)$ be the group of $\mathbb{F}_q$ -rational points of a connected reductive algebraic group G . Consider the set $\text{Irr}(G(\mathbb{F}_q))$ of complex irreducible representations of $G(\mathbb{F}_q)$. The classification of $\text{Irr}(G(\mathbb{F}_q))$ is a very old problem in representation theory, of fundamental importance. In [DL76], Deligne and Lusztig defined a finite set of representations (called *unipotent representations*)

$$\text{Unip}(G(\mathbb{F}_q)) \subset \text{Irr}(G(\mathbb{F}_q)).$$

These representations play a central role in the classification of $\text{Irr}(G(\mathbb{F}_q))$, which was later completed by Lusztig in [Lus84]. The importance of these representations is twofold:

- (i) They are classified by certain parameters related to complex nilpotent co-adjoint orbits (in particular, their classification is independent of q).
- (ii) The classification of $\text{Irr}(G(\mathbb{F}_q))$ reduces to the classification of $\text{Unip}(G(\mathbb{F}_q))$.

For details, we refer the reader to [Lus84].

Now, replace $\mathbb{F}_q$ with a local field k and let $G(k)$ be the group of k -rational points. Consider the set $\text{Irr}_u(G(k))$ of irreducible *unitary* representations of $G(k)$. The classification $\text{Irr}_u(G(k))$ is a classical problem in representation theory. Unlike its $\mathbb{F}_q$ -analog, it remains unsolved in general (for an introduction to the theory of unitary representations, we refer the reader to [Kna86]). An intriguing idea, with its origins in [BV85], is to find a finite set of representations (called *unipotent*, by analogy)

$$\text{Unip}(G(k)) \subset \text{Irr}_u(G(k))$$

that plays the same role with respect to $\text{Irr}_u(G(k))$ that $\text{Unip}(G(\mathbb{F}_q))$ plays with respect to $\text{Irr}(G(\mathbb{F}_q))$. In particular, these representations should satisfy suitable analogs of properties (i) and (ii) above.

The goal of this monograph is to give a definition of $\text{Unip}(G(k))$ in the case when $k = \mathbb{C}$. The definition we propose is geometric in nature, and well aligned with the orbit method, which is a guiding philosophy in the subject (see the preface above for a brief account or [Vog92] for a more detailed overview). Our

approach will shed some light on the more complicated case of $k = \mathbb{R}$; see Section 6.3.4.

In this monograph, we will take an algebraic point of view: our main objects of study are Harish-Chandra bimodules (rather than group representations, which are analytic in nature). It is well known that these two formalisms are equivalent. We recall the basic theory of Harish-Chandra bimodules in Section 3.4.

1.1 EXISTING CONSTRUCTIONS

In [BV85], Barbasch and Vogan, following ideas of Arthur [Art83], defined a set of representations called *special unipotent*. We will briefly review their construction and discuss its limitations.

Let G be a complex reductive algebraic group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{g}^\vee$ be the Langlands dual of $\mathfrak{g}$. If we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, there is a Cartan subalgebra $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$, canonically identified with $\mathfrak{h}^*$. To each nilpotent $\text{Ad}(\mathfrak{g}^\vee)$ -orbit $\mathbb{O}^\vee \subset (\mathfrak{g}^\vee)^*$, we can associate a maximal ideal in the enveloping algebra $U(\mathfrak{g})$ as follows. First, using an $\text{Ad}(\mathfrak{g}^\vee)$ -invariant isomorphism $(\mathfrak{g}^\vee)^* \simeq \mathfrak{g}^\vee$, we can identify $\mathbb{O}^\vee$ with a nilpotent $\text{Ad}(\mathfrak{g}^\vee)$ -orbit in $\mathfrak{g}^\vee$ (still denoted by $\mathbb{O}^\vee$). Choose an element $e^\vee \in \mathbb{O}^\vee$ and an $\mathfrak{sl}(2)$ -triple $(e^\vee, f^\vee, h^\vee)$. Acting by $\text{Ad}(\mathfrak{g}^\vee)$ if necessary, we can arrange so that $h^\vee \in \mathfrak{h}^\vee$. This element is well-defined modulo the usual action of W . Every W -orbit in $\mathfrak{h}^*$ determines an infinitesimal character for $U(\mathfrak{g})$ by means of the Harish-Chandra isomorphism $\mathfrak{Z}(\mathfrak{g}) \simeq \mathbb{C}[\mathfrak{h}^*]^W$. Consider the infinitesimal character corresponding to the element $\frac{1}{2}h^\vee \in \mathfrak{h}^\vee \simeq \mathfrak{h}^*$, and let $I_{\max}(\frac{1}{2}h^\vee) \subset U(\mathfrak{g})$ be the (unique) maximal ideal with this given infinitesimal character.

Definition 1.1 (Def 1.17, [BV85]). The *special unipotent ideal* attached to $\mathbb{O}^\vee \subset \mathfrak{g}^\vee$ is the maximal ideal $I_{\max}(\frac{1}{2}h^\vee) \subset U(\mathfrak{g})$. A *special unipotent bimodule* attached to $\mathbb{O}^\vee$ is an irreducible Harish-Chandra bimodule that is annihilated (on both sides) by $I_{\max}(\frac{1}{2}h^\vee)$. Denote the set of such bimodules by $\text{Unip}_{\mathbb{O}^\vee}^s(G)$.

There is a distinguished class of nilpotent orbits, first defined in [Lus79], called *special* nilpotent orbits. Below we will recall one of several equivalent definitions. If $\mathbb{O}^\vee$ is special, there is a well-known classification of $\text{Unip}_{\mathbb{O}^\vee}^s(G)$. Namely, there is a finite group $\overline{A}(\mathbb{O}^\vee)$ attached to $\mathbb{O}^\vee$ called the *Lusztig canonical quotient* (see [Lus84, Chp 13]) and a natural bijection

$$\text{Unip}_{\mathbb{O}^\vee}^s(G) \simeq \{\text{irreducible representations of } \overline{A}(\mathbb{O}^\vee)\}. \quad (1.1)$$

In the case when $\frac{1}{2}h^\vee$ is integral, this statement is contained in [BV85, Thm III]. The nonintegral case can be handled by similar methods; see [Won18b]. If $\mathbb{O}^\vee$ is *not* special, there is (to our knowledge) no known classification of $\text{Unip}_{\mathbb{O}^\vee}^s(G)$.

As an easy application of the main results in this monograph, we obtain one (see Remark 9.4). Our classification is of the same form as (1.1), except that $\overline{A}(\mathbb{O}^\vee)$ is replaced with the Galois group of a certain finite cover of a nilpotent coadjoint G -orbit.

Definition 1.1 has a number of limitations, which were known to Barbasch and Vogan at the time when [BV85] was written. Recall that, to every irreducible Harish-Chandra bimodule, there is an associated nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$ (see Section 3.1 for details). There is also an order-reversing map

$$D: \{\text{nilpotent orbits in } (\mathfrak{g}^\vee)^*\} \rightarrow \{\text{nilpotent orbits in } \mathfrak{g}^*\} \quad (1.2)$$

called Barbasch–Vogan–Lusztig–Spaltenstein (BVLS) duality (see Section 3.2). In [BV85] it is shown that the nilpotent orbit associated to a special unipotent bimodule $\mathcal{B} \in \text{Unip}_{\mathbb{O}^\vee}^s(G)$ is $D(\mathbb{O}^\vee) \subset \mathfrak{g}^*$.

The main limitation of Definition 1.1 is related to the fact that D is not (usually) surjective. Its image is the set of special nilpotent orbits (this explains the word “special” in “special unipotent”). If $\mathfrak{g} = \mathfrak{sl}(n)$, then D is a bijection and hence every orbit is special, but in all other types, there are nonspecial orbits. There are many bimodules attached to such orbits that deserve to be called unipotent, for example, the “metaplectic” bimodules for $\text{Sp}(2n)$; see Example 6.5(iv). Thus, Definition 1.1 is incomplete with respect to the problem posed in the previous section (as mentioned above, Barbasch and Vogan were well aware of this difficulty when they formulated their definition; see, e.g., the final paragraphs of [BV85, Section 1]. In fact, this issue is the main subject of the second half of [Vog87]).

Since the publication of [BV85], various attempts have been made to generalize the notion of “special unipotent.” Some examples include:

- Vogan’s weakly unipotent ideals ([Vog84]),
- Barbasch’s unipotent representations ([Bar89]),
- McGovern’s q -unipotent ideals ([McG94]), and
- Brylinski’s unipotent Dixmier algebras ([Bry03]).

None of these approaches is completely satisfactory. Vogan’s definition picks out an *infinite* set of ideals, which is inconsistent with the orbit method. The approaches of Barbasch and McGovern are based on ad hoc definitions that make sense only in classical types. Brylinski’s approach is appealingly geometric but applies only in special cases. Worse still, the approaches of Vogan, McGovern, and Brylinski allow for non-unitary bimodules (see Section 6.2 for examples).

1.2 VOGAN’S DESIDERATA

In [Vog87, Chps 6–12], Vogan lists some expected properties of unipotent bimodules. The most concrete of these expectations are catalogued below:

- (i) $\text{Unip}(G)$ is a finite subset of $\text{Irr}_u(G)$, attached in a natural fashion to G . See [Vog87, Chp 6, Problem I.2].
- (ii) The set $\text{Unip}(G)$ is partitioned into “packets,” $\text{Unip}_{\tilde{\mathbb{O}}}(G)$, parameterized by equivariant covers of nilpotent coadjoint G -orbits, i.e.,

$$\text{Unip}(G) = \bigsqcup_{\tilde{\mathbb{O}}} \text{Unip}_{\tilde{\mathbb{O}}}(G).$$

See [Vog87, “Def” 12.8].

- (iii) For each such $\tilde{\mathbb{O}}$, there is maximal, completely prime ideal $I(\tilde{\mathbb{O}}) \subset U(\mathfrak{g})$ such that

$$\text{LAnn}(\mathcal{B}) = \text{RAnn}(\mathcal{B}) = I(\tilde{\mathbb{O}}), \quad \forall \mathcal{B} \in \text{Unip}_{\tilde{\mathbb{O}}}(G).$$

See [Vog87, Conj 9.19] and [Vog87, “Def” 9.20].

- (iv) For every $\mathcal{B} \in \text{Irr}_u(G)$, there is a Levi subgroup $L \subset G$ and a unipotent bimodule $\mathcal{B}' \in \text{Unip}(L)$ such that $\mathcal{B}$ is obtained from $\mathcal{B}'$ through parabolic induction and/or a complementary series construction ([Vog87, Chp 6, Problem I.1]).
- (v) For every $\mathcal{B} \in \text{Unip}_{\tilde{\mathbb{O}}}(G)$ and $e \in \mathbb{O}$, there is a finite-dimensional representation χ of the component group G_e/G_e° such that

$$\mathcal{B} \simeq_G \text{AlgInd}_{G_e^\circ}^G \chi, \tag{1.3}$$

where $\text{AlgInd}_{G_e^\circ}^G$ denotes induction of algebraic group representations (see [Jan87, Sec 3.3]) and $\mathcal{B}$ is regarded as a G -representation via the adjoint action of $\mathfrak{g}$. See [Vog87, Requirement 11.23]. See also [Vog91, Conj 12.1] for a more precise criterion.

- (vi) $\text{Unip}(G)$ includes (as a proper subset) all special unipotent bimodules (cf. Definition 1.1). See the remarks following [Vog87, “Def” 12.8], as well as [Vog87, Chp 8].
- (vii) For each $\tilde{\mathbb{O}}$, the set $\text{Unip}_{\tilde{\mathbb{O}}}(G)$ is in one-to-one correspondence with irreducible representations of a finite group, which is related to the geometry of $\tilde{\mathbb{O}}$. See [Vog87, Chp 7] as well as [Vog87, Chp 10].

1.3 DEFINITIONS

The main contribution of this monograph is a definition of $\text{Unip}(G)$. Our definition is case-free, and geometric in nature. The main technical input is the theory of quantizations of conical symplectic singularities.

Conical symplectic singularities are a remarkable class of affine Poisson varieties with contracting $\mathbb{C}^\times$ -actions. Some basic examples include Kleinian singularities (i.e., varieties of the form $\mathbb{C}^2/\Gamma$, for a finite subgroup $\Gamma \subset \text{Sp}(2)$)

and the nilpotent cones of complex reductive Lie algebras. If X is a conical symplectic singularity, it makes sense to talk about its filtered quantizations (for a general discussion of graded Poisson varieties and their filtered quantizations, see Section 4.1). The classification of filtered quantizations of conical symplectic singularities was completed by the first-named author in [Los22a]. Roughly speaking, he shows that the set of filtered quantizations of X is in bijection with points in a complex vector space $\mathfrak{P}^X$, which is recovered from the geometry of X . A consequence of this result, which is of primary importance for our purposes, is that every conical symplectic singularity has a distinguished quantization, i.e., the one corresponding to $0 \in \mathfrak{P}^X$. We call this quantization the *canonical quantization* and denote it by $\mathcal{A}_0^X$. We conjecture that $\mathcal{A}_0^X$ is a simple algebra, for all X , and gather a great deal of evidence in Section 5.5.

Now let $\mathbb{O} \subset \mathfrak{g}^*$ be a nilpotent G -orbit and let $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$ be a finite connected G -equivariant cover. Note that $\tilde{\mathbb{O}}$ is a smooth symplectic variety, and its ring of regular functions $\mathbb{C}[\tilde{\mathbb{O}}]$ is a finitely generated algebra. Consider the affine variety $\tilde{X} := \text{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$. It is not difficult to show that $\tilde{X}$ is a conical symplectic singularity; (see [Los21, Lem 2.5]).

If $\mathcal{A}$ is *any* filtered quantization of $\tilde{X}$, the G -action on $\tilde{X}$ lifts to a G -action on $\mathcal{A}$ (by filtered algebra automorphisms). We show that the latter action is Hamiltonian, i.e., there is a quantum co-moment map $\Phi: U(\mathfrak{g}) \rightarrow \mathcal{A}$ (lifting the classical co-moment map $\mathbb{C}[\mathfrak{g}^*] \rightarrow \mathbb{C}[\tilde{\mathbb{O}}]$ induced from the natural map of varieties $\tilde{\mathbb{O}} \rightarrow \mathfrak{g}^*$). Note that Φ turns $\mathcal{A}$ into a Dixmier algebra, in the sense of [Vog90, Def 2.1]. Our main definition is as follows.

Definition 1.2. The *unipotent ideal* attached to $\tilde{\mathbb{O}}$ is the two-sided ideal

$$I_0(\tilde{\mathbb{O}}) := \ker(\Phi: U(\mathfrak{g}) \rightarrow \mathcal{A}_0^{\tilde{X}}).$$

A *unipotent bimodule* attached to $\tilde{\mathbb{O}}$ is an irreducible Harish-Chandra bimodule with left and right annihilators equal to $I_0(\tilde{\mathbb{O}})$. Denote the set of such bimodules by $\text{Unip}_{\tilde{\mathbb{O}}}(G)$.

It is easy to see that $I_0(\tilde{\mathbb{O}})$ is a completely prime primitive ideal with associated variety $\mathbb{O}$. We will see below that it is maximal (we supply a complete proof of this fact for linear classical groups. In the remaining cases, we provide only a sketch, leaving the details to a future paper [MBM23]). In fact, we will show that the maximality of $I_0(\tilde{\mathbb{O}})$ is equivalent to the simplicity of $\mathcal{A}_0^{\tilde{X}}$, which, as stated above, is a general expectation.

Much of this monograph is devoted to checking Vogan's desiderata for the definition above. Conditions (2), (5), (6), and (7) are verified for arbitrary G (Chapters 6 and 9). Conditions (1) and (3) are verified for linear classical G (Chapter 10 and Appendix B). Condition (4) is left more or less untouched (proving this condition is equivalent to the classification of $\text{Irr}_u(G)$ for arbitrary

G , a problem we do not solve in this monograph). Further motivation for Definition 1.2 will be provided in Section 6.3.

1.4 RESULTS

We now summarize the main results of this monograph.

1.4.1 Infinitesimal Characters and Maximality

In Chapter 8, we give a recipe for computing the infinitesimal characters of all unipotent ideals. There are two steps:

- (i) Reduce the calculation to *rigid* nilpotent orbits. This requires a fairly detailed analysis of the birational geometry of nilpotent covers.
- (ii) Determine the infinitesimal characters attached to rigid nilpotent orbits. For this step we rely heavily on the work of McGovern ([McG94]) and Premet ([Pre13]).

Following this recipe, one can, in principle, compute the infinitesimal character of an arbitrary unipotent ideal. For linear classical groups, we use this recipe to produce simple combinatorial formulas for unipotent infinitesimal characters; see Proposition 8.13. In ([MBM23]), we use a similar procedure to compute the unipotent infinitesimal characters for spin and exceptional groups.

If $I \subset U(\mathfrak{g})$ is a primitive ideal with known infinitesimal character and associated variety, there is a combinatorial condition for deciding whether I is a maximal ideal. We check this condition in many cases, and prove the following theorem.

Theorem 1.3 (Propositions B.1 and B.4 and Corollary B.14 below). *Suppose that G is a linear classical group, i.e., $G = \mathrm{SL}(n)$, $\mathrm{SO}(n)$, or $\mathrm{Sp}(2n)$. Let $\tilde{\mathbb{O}}$ be a G -equivariant cover of a nilpotent orbit in $\mathfrak{g}^*$. Then $I_0(\tilde{\mathbb{O}})$ is maximal.*

In [MBM23], we extend this result to arbitrary reductive groups.

1.4.2 Classification

In Chapter 6, we give geometric classifications of unipotent ideals and bimodules. First, we explain the classification of unipotent ideals. Let $\tilde{\mathbb{O}}$ and $\check{\mathbb{O}}$ be G -equivariant covers of a common nilpotent orbit $\mathbb{O} \subset \mathfrak{g}^*$. Consider the affine varieties $\tilde{X} = \mathrm{Spec}(\mathbb{C}[\tilde{\mathbb{O}}])$ and $\check{X} = \mathrm{Spec}(\mathbb{C}[\check{\mathbb{O}}])$. Both $\tilde{X}$ and $\check{X}$ contain finitely many G -orbits and $\tilde{\mathbb{O}}$ (resp. $\check{\mathbb{O}}$) is the unique open orbit in $\tilde{X}$ (resp. $\check{X}$). Further, every G -equivariant morphism $\tilde{\mathbb{O}} \rightarrow \check{\mathbb{O}}$ extends to a unique finite G -equivariant morphism $\tilde{X} \rightarrow \check{X}$. The latter morphism is étale over the open orbit $\tilde{\mathbb{O}} \subset \tilde{X}$.

Definition 1.4. A finite G -equivariant morphism $\tilde{X} \rightarrow \check{X}$ is *almost étale* if it is étale over the open subset in $\tilde{X}$ obtained by removing all G -orbits of codimension ≥ 4 .

Write $\tilde{\mathbb{O}} \geq \check{\mathbb{O}}$ if there is a G -equivariant morphism $\tilde{\mathbb{O}} \rightarrow \check{\mathbb{O}}$ such that the induced morphism $\tilde{X} \rightarrow \check{X}$ is almost étale. This defines a partial order on the set of G -equivariant covers of $\mathbb{O}$. Taking the symmetric closure, we get an equivalence relation on the same set. It is not difficult to see that each equivalence class contains a unique maximal element; see Lemma 6.27 below.

Theorem 1.5 (Theorem 6.29 below). *The ideal $I_0(\tilde{\mathbb{O}})$ depends only on the equivalence class of $\tilde{\mathbb{O}}$ with respect to the equivalence relation above. The passage from $\tilde{\mathbb{O}}$ to $I_0(\tilde{\mathbb{O}})$ defines a bijection between the following two sets:*

- *Equivalence classes of G -equivariant covers $\tilde{\mathbb{O}}$ of $\mathbb{O}$.*
- *Unipotent ideals with associated variety $\overline{\mathbb{O}}$.*

Next, we explain the classification of $\text{Unip}_{\tilde{\mathbb{O}}}(G)$. By Theorem 1.5, the set $\text{Unip}_{\tilde{\mathbb{O}}}(G)$ depends only on the equivalence class $[\tilde{\mathbb{O}}]$. Thus, we can assume for this discussion that $\tilde{\mathbb{O}}$ is maximal in $[\tilde{\mathbb{O}}]$. Let Π denote the automorphism group of $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$. This is a finite group, and can be described explicitly as follows. Choose an element $e \in \mathbb{O}$ and $x \in \tilde{\mathbb{O}}$ lying over e . Write G_e and G_x for the stabilizers in G of e and x , respectively. Then G_x is a subgroup of finite index in G_e and $\Pi \simeq N_{G_e}(G_x)/G_x$.

Theorem 1.6 (Theorem 6.31 below). *There is a natural bijection*

$$\{\text{irreducible representations of } \Pi\} \xrightarrow{\sim} \text{Unip}_{\tilde{\mathbb{O}}}(G).$$

1.4.3 Unitarity

The most basic requirement of unipotent bimodules is that they are unitary. We have some partial results in this direction.

Theorem 1.7 (Theorem 10.17 below). *Suppose that G is a linear classical group, and let $\tilde{\mathbb{O}}$ be a G -equivariant cover of a nilpotent orbit in $\mathfrak{g}^*$. Then every bimodule in $\text{Unip}_{\tilde{\mathbb{O}}}(G)$ is unitary.*

The proof is structured as follows. For classical $\mathfrak{g}$, there is a well-known classification of irreducible unitary Harish-Chandra bimodules, due to Barbasch ([Bar89]). The main result of his paper, namely, Theorem 12.8, is difficult to apply in general. An intermediate result in the same paper establishes the unitarity of a large collection of bimodules attached to *rigid* nilpotent orbits (see [Bar89, Prop 10.6]). As an easy consequence of this result, we show that all

unipotent bimodules attached to rigid orbits are unitary. To complete the proof of Theorem 1.7, we show that all of the remaining unipotent bimodules (i.e., those attached to covers of induced nilpotent orbits) are obtained from rigid ones by certain unitarity-preserving procedures, namely, parabolic induction and complementary series.

We note that a uniform proof of the unitarity of all unipotent bimodules was recently given in [DMB25].

1.4.4 G -Types

Assume that $\tilde{\mathbb{O}}$ is maximal in its equivalence class. Recall that the bimodules in $\text{Unip}_{\tilde{\mathbb{O}}}(G)$ are classified by an irreducible representation of Π (i.e., the automorphism group of the cover $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$).

Theorem 1.8 (Theorem 6.31 below). *Suppose that $I_0(\tilde{\mathbb{O}})$ is maximal, and let $\mathcal{B} \in \text{Unip}_{\tilde{\mathbb{O}}}(G)$. Let V denote the irreducible representation of Π corresponding to $\mathcal{B}$. Then there is a good filtration on $\mathcal{B}$ and an isomorphism of graded G -equivariant $S(\mathfrak{g})$ -modules*

$$\text{gr}(\mathcal{B}) \simeq (\mathbb{C}[\tilde{\mathbb{O}}] \otimes V)^{\Pi}.$$

The isomorphism of G -representations (1.3) predicted by Vogan is an easy consequence of this result (see Proposition 6.36 below).

1.4.5 “Special Unipotent” Implies “Unipotent”

In Chapter 9 we prove the following result.

Theorem 1.9 (Corollary 9.3 below). *Every special unipotent bimodule is unipotent.*

In fact, we prove a more precise result. To each nilpotent orbit $\mathbb{O}^{\vee} \subset (\mathfrak{g}^{\vee})^*$, we associate an equivalence class of covers $\tilde{\mathbb{D}}(\mathbb{O}^{\vee})$. This defines an injection

$$\begin{aligned} \tilde{\mathbb{D}} : \{\text{nilpotent orbits in } (\mathfrak{g}^{\vee})^*\} &\hookrightarrow \\ &\{\text{equivalence classes of covers of nilpotent orbits in } \mathfrak{g}^*\}, \end{aligned}$$

refining BVLS duality (1.2). We show that for each $\mathbb{O}^{\vee} \subset \mathfrak{g}^{\vee}$, the unipotent ideal $I_0(\tilde{\mathbb{D}}(\mathbb{O}^{\vee}))$ coincides with the special unipotent ideal $I_{\max}(\frac{1}{2}h^{\vee})$. Thus, $\text{Unip}_{\mathbb{O}^{\vee}}^s(G) = \text{Unip}_{\tilde{\mathbb{D}}(\mathbb{O}^{\vee})}(\tilde{G})$.

Our construction of $\tilde{\mathbb{D}}$ should be viewed as a special case of a more general (but still largely conjectural) duality known as *symplectic duality*; see [BLPW16b, Sec 10]. In Section 9.3, we will explain how our construction fits into this picture.

Remark 1.10. For $G = \mathrm{Sp}(2n)$, there is a notion, due to Barbasch, Ma, Sun, and Zhu, of a “metaplectic special” unipotent bimodule (see [BMSZ23]); these are irreducible bimodules with prescribed annihilators arising via a certain “metaplectic” modification of BVLS duality. It is easy to see that these bimodules are unipotent as well (compare to Chapter 9). Since we will not use this result, we omit the proof.

1.4.6 Dixmier Conjecture

In Section 7.8, we prove an old conjecture of Vogan (see [Vog90, Conj 2.3]). To each G -equivariant cover $\tilde{\mathbb{O}} \rightarrow \mathbb{O}$ of a coadjoint G -orbit $\mathbb{O} \subset \mathfrak{g}^*$ (not necessarily nilpotent) we attach a canonically defined Dixmier algebra $\mathrm{Dix}(\tilde{\mathbb{O}})$ such that

$$\mathbb{C}[\tilde{\mathbb{O}}] \simeq_G \mathrm{Dix}(\tilde{\mathbb{O}})$$

as representations of G . We prove the following result.

Theorem 1.11 (Theorem 7.54 below). *The assignment $\tilde{\mathbb{O}} \mapsto \mathrm{Dix}(\tilde{\mathbb{O}})$ defines an injective correspondence between G -equivariant covers of coadjoint orbits and Dixmier algebras for G .*

Moreover, if $\tilde{\mathbb{O}}$ is a *nilpotent* cover, then $\mathrm{Dix}(\tilde{\mathbb{O}})$ corresponds to the canonical quantization of $\mathbb{C}[\tilde{\mathbb{O}}]$. This is one of the principal justifications for Definition 1.2.

1.4.7 Discussion of Approach

Here we provide a brief description of our approach in this monograph, as it differs substantially from previous work in the subject. Whereas existing work in the area of unipotent representations has been largely Lie-theoretic, our approach in this monograph is mostly geometric. A recurring theme is that many interesting properties of unipotent bimodules that are very difficult to establish using Lie-theoretic methods become much more elementary and transparent from a geometric point of view. The most striking examples are Theorems 1.6 and 1.8.

Consider the problem of classifying unipotent bimodules. As discussed in Section 1.1, if $\mathbb{O}^\vee$ is special, the special unipotent bimodules attached to $\mathbb{O}^\vee$ are classified by irreducible representations of the Lusztig quotient $\bar{A}(\mathbb{O}^\vee)$ (this is essentially [BV85, Thm III]). The proof appearing in [BV85] is based on a complicated (and, in places, case-by-case) argument involving translation functors, Kazhdan-Lusztig cells, and the Springer correspondence. Even the definition of $\bar{A}(\mathbb{O}^\vee)$ is quite involved. Our approach, on the other hand, is geometric in nature. The group that replaces $\bar{A}(\mathbb{O}^\vee)$ is easy to define, our result is much more general, and our proof is more elementary. In the important special case

when $\mathbb{O}^\vee$ is even, the group we define is known to coincide with $\overline{A}(\mathbb{O}^\vee)$; this follows from [Los21, Proposition 7.4].

Another (related) example is Vogan’s conjecture (Theorem 1.8). Our proof of this result is easy and conceptual, whereas Lie-theoretic proofs, which exist in several cases, are difficult and ad hoc.

When it comes to results that are more Lie-theoretic in nature, our arguments become more technical. A good example is our (partial) computation of unipotent infinitesimal characters (Chapter 8). In the Barbasch–Vogan picture, special unipotent ideals are *defined* by their infinitesimal characters, which are given by the elegant formula $\frac{1}{2}h^\vee$. In our generalization, there is no such simple formula for unipotent infinitesimal characters, and lots of work is required to compute them. Similarly, in the Barbasch–Vogan picture, special unipotent ideals are *defined* to be maximal, whereas in our generalization lots of work is required to prove this. We note that the simplicity of the canonical quantization (Conjecture 5.14) is equivalent to the maximality of the unipotent ideal (Theorem 6.9) and seems to be a general phenomenon. However, proving the simplicity conjecture in general seems to be currently out of reach.

An interesting question is whether it is possible to come up with a simple *conceptual* description of the infinitesimal characters we obtain, perhaps in terms of the dual group $G^\vee$. It is possible that further developments in the area of symplectic duality will shed some light on this question.

1.5 OUTLINE OF MONOGRAPH

Here is an outline of the monograph. In Chapters 2, 3, and 4, we review some preliminary results that are needed in later chapters. Much of the material in these chapters already exists in the literature.

In Chapter 2 we collect some elementary facts about nilpotent orbits in complex semisimple Lie algebras and their (finite, connected) covers.

In Chapter 3 we recall some basic facts from Lie theory. This includes a discussion of primitive ideals in enveloping algebras, Harish-Chandra bimodules, Barbasch–Vogan–Lusztig–Spaltenstein (BVLS) duality, and $\mathcal{W}$ -algebras.

In Chapter 4 we review the key elements of the theory of conical symplectic singularities, their filtered quantizations, and Harish-Chandra bimodules over these filtered quantizations. This chapter contains two main results. The first is a classification of filtered quantizations of conical symplectic singularities, obtained by the first-named author in [Los22a]. To each conical symplectic singularity X we attach a finite-dimensional parameter space $\mathfrak{P}$. Filtered quantizations of X are parameterized by points in this parameter space, modulo a linear action of a certain finite group. The second main result is a classification of Harish-Chandra bimodules with full support over a filtered quantization of a conical symplectic singularity. In Section 4.13 we recall a category equivalence, first obtained in [Los21], between the category of Harish-Chandra bimodules with full support and the category of representations of a certain finite group.

In Sections 4.14 and 4.15 we provide a (partial) description of this finite group (see, in particular, Corollary 4.46 and Theorem 4.48).

In Chapter 5, we introduce the notion of a canonical quantization: the canonical quantization of a conical symplectic singularity is the filtered quantization corresponding to the zero parameter. If $\tilde{X} \rightarrow X$ is a finite Galois covering of conical symplectic singularities, we get a quantization of X by taking the algebra of invariants in the canonical quantization of $\tilde{X}$. This quantization of X , which is typically not canonical, will play an important role in our description of unipotent ideals attached to nontrivial nilpotent covers. In Chapter 5.3, we compute its quantization parameter. Also in this Chapter we compute the finite group that controls the category of Harish-Chandra bimodules with full support over a canonical quantization; see Proposition 5.3. In fact, we conjecture that canonical quantizations are simple algebras (and hence that *all* Harish-Chandra bimodules are automatically of full support). This conjecture is discussed in Section 5.5.

In Chapter 6 we introduce our main objects of study: unipotent ideals and unipotent bimodules. These objects are closely related to canonical quantizations, so the results of Chapter 5 are applicable. Our primary goal in Chapters 6–10 is to show that these distinguished ideals and bimodules have all (or at least many) of the properties one would hope for—in other words, to convince the reader that our definitions are the right ones. As a first step toward this goal, we show in Chapter 6 that our unipotent ideals and bimodules have geometric classifications. Unipotent ideals are classified by equivalence classes of covers with respect to a geometrically defined equivalence relation connected to codimension 2 singularities. Unipotent bimodules are classified by irreducible representations of certain finite groups. The latter classification bears an encouraging resemblance to the Barbasch–Vogan classification of special unipotent bimodules.

One of the most intriguing questions that arises from the results in this monograph is whether our notion of “unipotent” can be meaningfully extended to all real reductive Lie groups. We discuss this question briefly in Section 6.3.4. One can naively extend our definition to representations of a (non-complex) Lie group by requiring that the annihilator of the underlying $(\mathfrak{g}, K)$ -module is a unipotent ideal. Some very simple examples show that this naive definition is flawed—many of the representations in question are non-unitary. Remarkably, this issue seems to vanish if $\tilde{\mathbb{O}} = \mathbb{O}$ is birationally rigid.

In Chapters 7–10, we establish some additional key properties of unipotent ideals and bimodules. Unlike the results in Chapter 6, these require a detailed understanding of the infinitesimal characters of unipotent ideals. This turns out to be a rather delicate matter. In Chapters 7 and 8 we present an algorithm for computing the infinitesimal character $\gamma_0(\tilde{\mathbb{O}})$ of the unipotent ideal attached to an arbitrary nilpotent cover. The main formula is stated in Chapter 8: for each nilpotent cover $\tilde{\mathbb{O}}$, there is a Levi subgroup $L \subset G$ and a rigid L -orbit $\mathbb{O}_L$ such that

$$\gamma_0(\tilde{\mathbb{O}}) = \gamma_0(\mathbb{O}_L) + \delta,$$

where δ is a small “shift” related to the quantization parameter constructed in Chapter 5. Thus, the computation of $\gamma_0(\mathbb{O})$ is a two-step process: first, compute the infinitesimal character attached to the rigid nilpotent orbit $\mathbb{O}_L$, and then compute the element δ . For the first step, we appeal to known results of McGovern (for $\mathfrak{g}$ a classical Lie algebra) and Premet (for $\mathfrak{g}$ exceptional). There are six “bad” orbits in exceptional types that are not handled by Premet (we handle these orbits separately in Appendix C—they are the most difficult cases). On the other hand, computing the element δ requires a fairly detailed analysis of the birational geometry of $\mathbb{O}$. The required machinery is developed in Chapter 7.

Using our computation of unipotent infinitesimal characters, we prove in Section 8.5 that all unipotent ideals for linear classical groups are maximal. The argument relies on some rather tedious combinatorics, which is relegated to Appendix B.

In Chapter 9 we introduce our refined BVLS duality. This is a map that takes nilpotent orbits for the Langlands dual group $G^\vee$ to nilpotent covers for G . One consequence of this duality is that all special unipotent ideals (in the sense of [BV85]) are unipotent.

In Chapter 10, we turn our attention to the unitarity of unipotent bimodules and the related issue of their construction via parabolic induction. The main result in this chapter, Theorem 10.17, is that all unipotent bimodules for linear classical groups are unitary. The proof has two components: first, we show that all unipotent bimodules attached to rigid orbits are unitary. Then, we show that these “rigid” unipotents generate all others (through several unitarity-preserving operations). The first step is an immediate consequence of an old result of Barbasch ([Bar89]) and our infinitesimal character computations. For the second step, we introduce a version of parabolic induction for Harish-Chandra bimodules over filtered quantizations. Under suitable conditions, this construction coincides with the usual induction of Harish-Chandra bimodules (this is proved in Appendix A), and thus preserves unitarity.

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