Contents

Introduction

0.1 FORMAL-ANALYTIC SURFACES AS ANALOGUES OF FORMAL SURFACES OR GERMS OF ANALYTIC SURFACES ALONG A PROJECTIVE CURVE

0.1.1

This memoir is devoted to the study of *formal-analytic arithmetic surfaces* and to diverse applications of these to the geometry of projective and quasi-projective arithmetic surfaces.

Formal-analytic arithmetic surfaces have been introduced in [Bos20, Section 10.6], to provide a natural geometric framework to the algebraization theorems of the Chudnovskys in [CC85a] and [CC85b], of André (see [And04] for exposition and references), and of their developments in [Bos01] and [BCL09]. Special instances of formal-analytic arithmetic surfaces also occur implicitly in the recent paper [CDT25] by Calegari, Dimitrov, and Tang, and their work has been an important inspiration for the authors of this memoir.[1] In turn, some of the ideas in this memoir, including the numerical improvements obtained here via the introduction of the "overflow" invariant, have been applied to the spectacular irrationality results of [CDT24].

At least implicitly, formal-analytic arithmetic surfaces have been considered in various other contexts: notably in the famous note [Bor94] by E. Borel, in the work of Harbater [Har84, Har88], and in the theory of Eisenstein series associated to loop groups over the integers, as developed by Garland and Patnaik in [GP08].[2]

Moreover, formal-analytic arithmetic surfaces are closely related to Berkovich spaces over $\mathbb{Z}$, as studied by Poineau [Poi10], [Poi13] and Lamanissier and Poineau [LP20]. They also constitute a natural ground for applying the new techniques of analytic geometry currently developed by Clausen and Scholze [CS22] in the framework of condensed mathematics, even though we will not attempt to do so here.

0.1.2

The point of view developed in this memoir is, first, that formal-analytic arithmetic surfaces are interesting objects in themselves, which admit non-trivial global invariants, and, second, that these invariants constitute a natural tool to investigate classical questions of arithmetic geometry.

Formal-analytic arithmetic surfaces constitute arithmetic counterparts of germs $\mathcal{V}$ of complex analytic surfaces along a projective complex curve C, or in a more

[1] Understanding the relation between the arithmetic holonomicity theorem in [CDT25] and their earlier results on formal-analytic arithmetic surfaces has been a major incentive for the authors to establish the main result in Chapter 5 concerning the "overflow" invariant, Theorem 5.4.1.

[2] See the recent work by Dutour and Patnaik [DP22] for new developments and additional references.

algebraic context, of two-dimensional smooth formal k-schemes $\widehat{\mathcal{V}}$ with scheme of definition a projective curve C over some field k.

As made clear by the classical work of Grauert on modifications [Gra62] and its application to singularities of surfaces [Lau71], or by the work of Artin on contractions [Art70], these germs of complex analytic surfaces and these formal surfaces naturally arise in the study of algebraic surfaces and two-dimensional algebraic spaces. Similarly, formal-analytic arithmetic surfaces are natural tools to investigate the geometry *à la Arakelov* of arithmetic surfaces, that is, of two-dimensional integral quasi-projective flat schemes over $\mathbb{Z}$.

More specifically, a major theme of this monograph is how one may obtain information about projective and quasi-projective arithmetic surfaces and their étale fundamental groups by studying the geometry of certain formal-analytic arithmetic surfaces mapping naturally to them—especially when the morphism is not an immersion. Such morphisms appear naturally in several context, perhaps most notably when looking at the geometry of a modular curve near a cusp, as exemplified in [CDT25] and 9.3.4 below.

With this them in mind, we establish diverse results concerning morphisms between arithmetic surfaces and their fundamental groups which illustrate this philosophy. These arithmetic results may be seen as arithmetic counterparts of some of the results of Nori in [Nor83] concerning quasi-projective complex surfaces.

The germs of analytic surfaces $\mathcal{V}$ along a projective complex curve C that appear in [Nor83] satisfy a *pseudoconcavity* condition—basically a positivity condition on the normal bundle $N_C\mathcal{V}$ or on the self-intersection $C \cdot C$ of C in $\mathcal{V}$—while in the work of Grauert and Artin on contractions, they satisfy a *pseudoconvexity* condition, related to the negativity of $N_C\mathcal{V}$ and $C \cdot C$. The dichotomy pseudoconvex/pseudoconcave turns out to govern also the arithmetic geometry of formal-analytic arithmetic surfaces.

0.1.3

To put it briefly, our aim in this memoir is to demonstrate that formal-analytic arithmetic surfaces admit a non-trivial geometry, relevant for the study of classical objects of arithmetic geometry—including notably quasi-projective arithmetic surfaces and their étale fundamental groups. This geometry involves *global real-valued invariants*, as in the theory of heights and Arakelov geometry. These invariants are suitably defined *arithmetic intersection numbers*, and *θ-invariants* of possibly infinite-dimensional Hermitian vector bundles, as defined in [Bos20].

To achieve this aim without too lengthy foundational preliminaries, we have deliberately limited the generality of the class of formal-analytic arithmetic surfaces investigated in this memoir, by sticking to the class already introduced in [Bos20, Chapter 10].[3]

[3]In 0.2.3, we give some indications on a wider class of formal-analytic arithmetic surfaces to which most of our results extend.

Our objectives have rather been (i) to clarify the geometric meaning of our constructions, notably by discussing in detail a series of results in complex analytic and algebraic geometry of which our main results are arithmetic counterparts, (ii) to spell out a few "concrete" consequences of our general finiteness results concerning pseudoconcave formal-analytic arithmetic surfaces that may be formulated in elementary terms, and (iii) to emphasize the role of a new Archimedean invariant—the "overflow" attached to a complex analytic map from a pointed compact Riemann with boundary to another Riemann surface—which naturally arises when investigating the morphisms from formal-analytic to quasi-projective arithmetic surfaces.

The scope of this monograph is already wide enough to provide a geometric reformulation of the arithmetic holonomicity theorem of [CDT25] by framing it as a consequence of our finiteness results associated to morphisms from a pseudoconcave formal-analytic arithmetic surface to a quasi-projective arithmetic surface. As discussed in 8.3.2 below, the numerical hypotheses of the holonomicity theorem correspond to a pseudoconcavity condition for a specific formal-analytic surface obtained by gluing a closed disk to a formal scheme, and our finiteness result, through the overflow mentioned above, gives an improved and more symmetric form of the "holonomy bounds" that they prove.

0.2 FORMAL-ANALYTIC SURFACES OVER Spec $\mathbb{Z}$: DEFINITION

In this introduction, we present some of our main results concerning formal-analytic arithmetic surfaces, focusing on the simple case of formal-analytic arithmetic surfaces over Spec $\mathbb{Z}$.

0.2.1

We begin by introducing the main character of this memoir. A *smooth formal-analytic arithmetic surface over* Spec $\mathbb{Z}$ is defined as a triple:

$$\widetilde{\mathcal{V}} = (\widehat{\mathcal{V}}, (V, O), \iota),$$

where

- $\widehat{\mathcal{V}}$ is a formal scheme, isomorphic to $\mathrm{Spf}\,\mathbb{Z}[[X]]$;
- V is a connected compact Riemann surface with non-empty boundary ∂V, equipped with a real structure,[4] and O is a real point[5] in the interior $\mathring{V}$ of V;
- ι is an isomorphism of complex formal curves, compatible with the real structures:

$$\iota : \widehat{\mathcal{V}}_{\mathbb{C}} \xrightarrow{\sim} \widehat{V}_O.$$

The gluing data provided by the isomorphism ι may be described in more elementary terms as follows.

[4] That is, an antiholomorphic involution c. We shall call it "complex conjugation."

[5] That is, a fixed point of the complex conjugation c.

We may choose an analytic coordinate z on some open neighborhood of O in V that is compatible with complex conjugation.[6] This coordinate establishes an isomorphism of complex formal curves:

$$\widehat{V}_O \xrightarrow{\sim} \operatorname{Spf} \mathbb{C}[[z]].$$

Moreover, the isomorphism:

$$\widehat{\mathcal{V}} \xrightarrow{\sim} \operatorname{Spf} \mathbb{Z}[[X]]$$

induces an identification:

$$\widehat{\mathcal{V}}_{\mathbb{C}} \xrightarrow{\sim} \operatorname{Spf} \mathbb{C}[[X]].$$

Therefore the data of the isomorphism ι is equivalent to the one of a formal series $\psi \in \mathbb{R}[[X]]$ such that:

$$\psi(0) = 0 \quad \text{and} \quad \psi'(0) \neq 0; \tag{0.2.1}$$

namely to the isomorphism:

$$\operatorname{Spf} \mathbb{C}[[X] \simeq \widehat{\mathcal{V}}_{\mathbb{C}} \overset{\iota}{\xrightarrow{\sim}} \widehat{V}_O \simeq \operatorname{Spf} \mathbb{C}[[z]],$$

compatible with the real structures, is associated the formal series:

$$\psi := \iota^* z.$$

In particular, when the Riemann surface with boundary V is a closed disk—say when the pair (V, O) is $(\overline{D}(0;1), 0)$—we may take the standard coordinate $z : \overline{D}(0;1) \hookrightarrow \mathbb{C}$ as the analytic coordinate near O in V, and we may associate a smooth formal-analytic arithmetic surface over $\operatorname{Spec} \mathbb{Z}$ to every formal series $\psi \in \mathbb{R}[[X]]$ satisfying (0.2.1) (see 0.3.4 below).

This discussion shows that formal-analytic arithmetic surfaces over $\operatorname{Spec} \mathbb{Z}$ are easily constructed, "soft" mathematical objects, and one may wonder whether they are worthy of interest.

0.2.2

A first positive answer to this question is that, as already mentioned above, formal-analytic arithmetic surfaces arise naturally as the counterparts, in the dictionary between number fields and function fields, of the germs of analytic surfaces—or of the formal surfaces—fibered over a projective curve.

Let us explain this in more detail.

0.2.2.1

Let C be a smooth connected projective curve, say over the complex field $\mathbb{C}$, and let Σ be a non-empty finite subset of C. The complement

[6] Namely it satisfies $z \circ c = \overline{z}$.

$$\mathring{C} := C \setminus \Sigma$$

is an affine curve, and its ring of regular functions $\mathcal{O}(\mathring{C})$ is a Dedekind ring.

In the classical analogy between number fields and function fields, the ring of integers $\mathcal{O}_K$ of a number field K (resp. the scheme $\operatorname{Spec} \mathcal{O}_K$) is seen as the arithmetic counterpart of the ring $\mathcal{O}(\mathring{C})$ (resp. of the smooth affine curve $\mathring{C}$), the set of Archimedean places of K as the counterpart of Σ, the Hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$ as the counterparts of the vector bundles over C, the (real-valued) Arakelov degree of these Hermitian vector bundles as the counterpart of the (integer-valued) degree of vector bundles over C, etc.

This analogy is pursued much further in Arakelov geometry, where a regular projective scheme $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}_K$, with the projective complex manifolds[7] $(\mathcal{X}_\sigma(\mathbb{C}))_{\sigma: K \hookrightarrow \mathbb{C}}$ endowed with suitable Kähler structures, appears as the counterpart of a smooth projective variety X fibered over C.

0.2.2.2

The formal-analytic arithmetic surfaces investigated in this memoir constitute a new entry in the dictionary relating number fields and function fields. Their function field analogues are the following geometric objects.

Consider a smooth connected complex analytic surface $\mathcal{V}$, a surjective (necessarily flat) complex analytic map with connected fibers:

$$\pi_{\mathcal{V}} : \mathcal{V} \longrightarrow C$$

and a complex analytic section of $\pi_{\mathcal{V}}$:

$$\varepsilon : C \longrightarrow \mathcal{V}.$$

Assume, moreover, that, for every $x \in \Sigma$, the connected curve $\pi_{\mathcal{V}}^{-1}(x)$ is noncompact, and that we are given a reduced, compact, connected curve F_x in $\pi_{\mathcal{V}}^{-1}(x)$ containing $\varepsilon(x)$. Then the divisor

$$D := \varepsilon(C) + \sum_{x \in \Sigma} F_x$$

is compact and connected, and we may consider the germ $\mathcal{V}_D^{\mathrm{an}}$ of complex analytic surface of $\mathcal{V}$ along D. It is "fibered over C," in the sense that it is equipped with the (germ of) analytic map:

$$\pi_{\mathcal{V} | \mathcal{V}_D^{\mathrm{an}}} : \mathcal{V}_D^{\mathrm{an}} \longrightarrow C. \tag{0.2.2}$$

In the dictionary between number fields and function fields, the smooth formal-analytic surfaces over $\operatorname{Spec} \mathcal{O}_K$ investigated in this memoir correspond to the germs

[7] We denote by $\sigma : K \hookrightarrow \mathbb{C}$ the field embeddings of K in $\mathbb{C}$. Their classes up to complex conjugation are in bijection with the Archimedean places of K. By $\mathcal{X}_\sigma$, we denote the complex projective variety deduced from the $\mathcal{O}_K$-scheme $\mathcal{X}$ by the base change $\sigma : \mathcal{O}_K \to \mathbb{C}$.

of complex analytic surfaces $\mathcal{V}_D^{\mathrm{an}}$ equipped with the map (0.2.2)—or to a formal variant of these, where germs of complex analytic surfaces along the compact divisor D are replaced by formal surfaces admitting D as scheme of definition.[8]

0.2.2.3

The following remarks should clarify this correspondence.

The germ of surface $\mathcal{V}_D^{\mathrm{an}}$ along D may be seen as the "union" of the germ $\mathcal{V}^{\mathrm{an}}_{\varepsilon(\mathring{C})}$ of $\mathcal{V}$ along the affine curve $\mathring{C}$, and of its germs $\mathcal{V}_{F_x}^{\mathrm{an}}$ along the vertical divisors F_x for $x \in \Sigma$, glued along the "intersections:"

$$\mathcal{V}^{\mathrm{an}}_{\varepsilon(\mathring{C})} \cap \mathcal{V}^{\mathrm{an}}_{F_x}. \tag{0.2.3}$$

In the above definition of smooth formal-analytic surfaces over $\operatorname{Spec}\mathbb{Z}$, the "algebraic" or "formal" data:

$$\widehat{\mathcal{V}} \simeq \operatorname{Spf}\mathbb{Z}[[X]]$$

and its structure map:

$$\widehat{\mathcal{V}} \longrightarrow \operatorname{Spec}\mathbb{Z}$$

play the role of $\mathcal{V}^{\mathrm{an}}_{\varepsilon(\mathring{C})}$ and of the restriction:

$$\pi_{\mathcal{V}|\varepsilon(\mathring{C})} : \mathcal{V}^{\mathrm{an}}_{\varepsilon(\mathring{C})} \longrightarrow \mathring{C}.$$

The "analytic data" (V, O) play the role of $(\mathcal{V}^{\mathrm{an}}_{F_x}, \varepsilon_x)$, where ε_x denotes the germ of ε at x. Finally, the isomorphism ι corresponds to the "gluing isomorphism" along (0.2.3).

0.2.3

In this monograph, we work more generally with *smooth formal-analytic surfaces over* $\operatorname{Spec} \mathcal{O}_K$, where we denote by K a number field, and by $\mathcal{O}_K$ its ring of integers. A still more general—and arguably more natural—framework would have been the one of *regular formal-analytic surfaces over* $\operatorname{Spec}\mathbb{Z}$. Those are defined as triples $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, V, \iota)$, where:

- $\widehat{\mathcal{V}}$ is a regular noetherian formal scheme of pure dimension 2, whose scheme of definition $|\widehat{\mathcal{V}}|$ is proper over $\operatorname{Spec}\mathbb{Z}$, of pure dimension 1;
- V is a compact Riemann with boundary, equipped with a real structure;
- $\iota : \widehat{\mathcal{V}}_{\mathbb{C}} \to \mathring{V}$ is an embedding, compatible with complex conjugation, of the formal complex curve $\widehat{\mathcal{V}}_{\mathbb{C}}$ into the interior $\mathring{V}$ of V.

Many constructions and results in this memoir actually extend to this setting, provided $\widetilde{\mathcal{V}}$ satisfies a natural connectedness condition, when moreover the following conditions are satisfied:

[8]This formal variant makes sense, not only over the complex field, but over an arbitrary base field.

- every connected component of V has a non-empty boundary;[9]
- for every prime p, every connected component of the formal scheme $\widehat{\mathcal{V}}_{\mathbb{F}_p}$ over $\mathbb{F}_p$ is *not* a scheme.

However these extensions require further foundational developments concerning formal schemes and their intersection theory, and we defer them to some future work.

0.3 FORMAL-ANALYTIC SURFACES OVER Spec $\mathbb{Z}$: FURTHER DEFINITIONS AND MAIN RESULTS

The above dictionary between number fields and function fields extends naturally to various geometric objects involving germs of analytic surfaces fibered over C, such as vector bundles or morphisms to algebraic varieties.

The translation procedure for these diverse notions follows the same pattern as in 0.2.2.3. Their arithmetic analogues are defined in terms of (i) an algebraic or formal part (that corresponds to their restriction over $\mathring{C}$ in the geometric context), (ii) an analytic part (that corresponds to their restriction over the germs in C of x in the finite set Σ that plays the role of Archimedean places), and (iii) some additional gluing data. In particular, as in "classical" Arakelov geometry, the arithmetic counterpart of vector bundles are Hermitian vector bundles over formal-analytic arithmetic surfaces.

At this stage, the possibility of such a translation is hardly surprising. It is more remarkable that some basic notions of intersection theory on (possibly non-compact) complex analytic surfaces may be translated to formal-analytic arithmetic surfaces, in the spirit of Arakelov intersection numbers as defined in [Del87] and [GS90].

For instance, if L is a (germ of) analytic line bundle over $\mathcal{V}_D^{\mathrm{an}}$ and if Z is a divisor supported on D, we may define the intersection number $L \cdot Z$ since $|Z|$ is compact. Similarly, in the arithmetic setting, we may define the (real-valued) Arakelov intersection number $\overline{L} \cdot Z$ of a Hermitian line bundle $\overline{L}$ over a formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ and of a suitably defined Arakelov divisor with compact support Z on $\widetilde{\mathcal{V}}$.

As indicated in 0.1.2 above, our main results in this memoir concern formal-analytic surfaces $\widetilde{\mathcal{V}}$ that satisfy a suitable pseudoconcavity condition. In this case, we shall firstly show that the "spaces of global sections" of Hermitian vector bundles over $\widetilde{\mathcal{V}}$ satisfy some finiteness property, and that their "absolute dimension" defined in terms of the θ-invariants constructed in the monograph [Bos20] satisfies remarkable estimates in terms of the Arakelov intersection numbers mentioned above.

Secondly, we will attach some intersection-theoretic invariants to morphisms from formal-analytic arithmetic surfaces to "classical" quasi-projective arithmetic surfaces, and demonstrate their relevance to various questions involving quasi-projective arithmetic surfaces and their étale fundamental groups.

In this section, we give a sample of these results, which constitute a second positive answer to our previous question at the end of 0.2.1 concerning the significance of the notion of formal-analytic arithmetic surface.

[9]The boundary being non-empty makes it possible to use the theory of Green functions and equilibrium potentials.

For simplicity, we focus on the case of formal-analytic surfaces over $\operatorname{Spec}\mathbb{Z}$. We have tried to present precise and significant statements, without assuming any prior knowledge of Arakelov geometry, of potential theory, or of the θ-invariants introduced in [Bos20]. Hopefully the self-contained character of this section will constitute an excuse for the terseness of its presentation.

We denote by $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, (V, O), \iota)$ a smooth formal-analytic arithmetic surface over $\operatorname{Spec}\mathbb{Z}$, as defined in 0.2.1.

The structure map of $\widehat{\mathcal{V}}$:

$$\widehat{\mathcal{V}} \simeq \operatorname{Spf}\mathbb{Z}[[X]] \longrightarrow \operatorname{Spec}\mathbb{Z} \tag{0.3.1}$$

defines an isomorphism:

$$|\widehat{\mathcal{V}}| \xrightarrow{\sim} \operatorname{Spec}\mathbb{Z}.$$

Its inverse defines a section (actually the unique section) of (0.3.1), which we shall denote by:

$$P : \operatorname{Spec}\mathbb{Z} \longrightarrow \widehat{\mathcal{V}}.$$

We shall also denote by P its image, that is the scheme of definition $|\widehat{\mathcal{V}}|$ of $\widehat{\mathcal{V}}$.

0.3.1

If X is an arithmetic scheme, namely a separated scheme of finite type over $\operatorname{Spec}\mathbb{Z}$, we define a morphism:

$$\alpha : \widetilde{\mathcal{V}} \longrightarrow X$$

as a pair:

$$\alpha := (\widehat{\alpha}, \alpha^{\mathrm{an}}),$$

where:

$$\widehat{\alpha} : \widehat{\mathcal{V}} \longrightarrow X$$

is a morphism of (formal) schemes, and where:

$$\alpha^{\mathrm{an}} : V \longrightarrow X(\mathbb{C})$$

is a complex analytic map[10] such that the following compatibility relation is satisfied:

$$\widehat{\alpha}_{\mathbb{C}} = \widehat{\alpha}^{\mathrm{an}} \circ \iota. \tag{0.3.2}$$

In (0.3.2), we have denoted by:

$$\widehat{\alpha}_{\mathbb{C}} : \widehat{\mathcal{V}}_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$$

the morphism of complex (formal) schemes deduced from $\widehat{\alpha}$ by the base change $\mathbb{Z} \hookrightarrow \mathbb{C}$, and by

[10] Analytic up to the boundary ∂V of the Riemann surface with boundary V.

$$\widehat{\alpha}^{\mathrm{an}} : \widehat{V}_O \longrightarrow X_{\mathbb{C}}$$

the formal germ of α^{an} at O.

For instance, a morphism: $f : \widetilde{\mathcal{V}} \to \mathbb{A}^1_{\mathbb{Z}}$ is a pair $(\widehat{f}, f^{\mathrm{an}})$, where $\widehat{f}$ is an element of $\Gamma(\widehat{V}, \mathcal{O}_{\widehat{V}}) \simeq \mathbb{Z}[[T]]$, and f^{an} an element of $\Gamma(V, \mathcal{O}^{\mathrm{an}}_V)$—that is, a complex analytic function on V, analytic up to the boundary—that satisfy the compatibility relation:

$$\widehat{f}_{\mathbb{C}} = \widehat{f^{\mathrm{an}}} \circ \iota. \qquad (0.3.3)$$

The set of these morphisms from $\widetilde{\mathcal{V}}$ to $\mathbb{A}^1_{\mathbb{Z}}$ defines the $\mathbb{Z}$-algebra $\mathcal{O}(\widetilde{\mathcal{V}})$ of *regular functions on* $\widetilde{\mathcal{V}}$.

We may similarly define the field $\mathcal{M}(\widetilde{\mathcal{V}})$ of *meromorphic functions on* $\widetilde{\mathcal{V}}$, which is a field extension of $\mathbb{Q}$. Its elements are the pairs $f := (\widehat{f}, f^{\mathrm{an}})$ where $\widehat{f}$ is a formal meromorphic function on $\widehat{V}$—or equivalently an element of the fraction field $\operatorname{Frac}\mathbb{Z}[[T]]$ of $\mathbb{Z}[[T]]$—and f^{an} is a meromorphic function on V (defined up to the boundary) such that the compatibility condition (0.3.3) is satisfied.

Vector bundles and *Hermitian vector bundles* over $\widetilde{\mathcal{V}}$ are defined by "gluing" a vector bundle over $\widehat{\mathcal{V}}$ and a vector bundle or a Hermitian vector bundle over the Riemann surface V. Namely a vector bundle (resp. a Hermitian vector bundle) over $\widetilde{\mathcal{V}}$ is the data:

$$E := (\widehat{E}, E^{\mathrm{an}}, \varphi) \qquad (\text{resp. } \overline{E} := (\widehat{E}, E^{\mathrm{an}}, \varphi, \|.\|))$$

of a vector bundle $\widehat{E}$ over $\widehat{\mathcal{V}}$, of a complex analytic vector bundle E^{an} over V, and of an isomorphism of vector bundles over the complex formal curve $\widehat{\mathcal{V}}_{\mathbb{C}}$:

$$\varphi : \widehat{E}_{\mathbb{C}} := \widehat{E} \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} \iota^*(E^{\mathrm{an}}_{\widehat{V}_O}),$$

(resp. and of some C^∞ Hermitian metric $\|.\|$ on the vector bundle E^{an} over V). These data are assumed to be compatible with complex conjugation.

0.3.2

It is possible to develop a version of Arakelov intersection theory on a formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ as above. In spite of its rudimentary character, this arithmetic intersection theory will allow us to associate some significant invariants to formal-analytic arithmetic surfaces and to their morphisms with values in a "classical" quasi-projective arithmetic surface—namely, in an integral arithmetic scheme of dimension 2, quasi-projective and flat over $\operatorname{Spec}\mathbb{Z}$.

The main analytic tool for developing an Arakelov intersection theory on the formal-analytic arithmetic surface $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, (V, O), \iota)$ is the notion of a *Green function* for the point O in V that satisfies the Dirichlet boundary condition. By definition, these Green functions are the real-valued C^∞ functions g on $V \setminus \{O\}$ satisfying:

$$g_{|\partial V} = 0, \qquad (0.3.4)$$

that admit a logarithmic singularity at O. This last condition means that, if z is a local analytic coordinate on some open neighborhood U of O in V, there exists $h \in C^\infty(U, \mathbb{R})$ such that:

$$g = \log |z - z(O)|^{-1} + h \quad \text{on } U \setminus \{O\}. \tag{0.3.5}$$

If g is a Green function as above, invariant under the complex conjugation of V, then the pair (P, g) may be seen as a compactly supported Arakelov divisor on $\widetilde{\mathcal{V}}$.

If moreover $\overline{L} := (\widehat{L}, L^{\text{an}}, \varphi, \|.\|)$ is a Hermitian line bundle over $\widetilde{\mathcal{V}}$, we may define the *height* of P with respect to $\overline{L}$ as the Arakelov degree:

$$\text{ht}_{\overline{L}}(P) := \widehat{\deg}\, P^*\overline{L} \in \mathbb{R} \tag{0.3.6}$$

and the *arithmetic intersection number of* $\overline{L}$ *and the Arakelov divisor* (P, g) as the sum:

$$\overline{L} \cdot (P, g) := \text{ht}_{\overline{L}}(P) + \int_V g\, c_1(\overline{L}_{\mathbb{C}}) \in \mathbb{R}. \tag{0.3.7}$$

The definitions (0.3.6) and (0.3.7) are similar to well-known definitions concerning heights and Arakelov intersection numbers on "classical" projective arithmetic surfaces.

In the right-hand side of (0.3.6), $P^*\overline{L}$ is the Hermitian line bundle over $\operatorname{Spec} \mathbb{Z}$ defined by the free $\mathbb{Z}$-module of rank one $P^*\widehat{L}$ and by the norm $\|.\|_O$ on the complex line:

$$(P^*L)_{\mathbb{C}} \xrightarrow[\sim]{\varphi_{P_{\mathbb{C}}}} L^{\text{an}}_{|O}.$$

Moreover, $\widehat{\deg}\, P^*\overline{L}$ denotes the Arakelov degree of this Hermitian line bundle. If s denotes a generator of the $\mathbb{Z}$-module $P^*\widehat{L}$, it is defined as:

$$\widehat{\deg}\, P^*\overline{L} := \log \|s\|_O^{-1}.$$

In the right-hand side of (0.3.7), we denote by $c_1(\overline{L}_{\mathbb{C}})$ the first Chern form of the Hermitian line bundle $(L^{\text{an}}, \|.\|)$ on V, defined by:

$$c_1(\overline{L})_{|U} := (2\pi i)^{-1} \partial\overline{\partial} \log \|s\|^2,$$

where s is a non-vanishing complex analytic section of L over some open subset U of V.

0.3.3

Among the Green functions for the point O in V, defined by conditions (0.3.4) and (0.3.5), there is a distinguished one, namely the *equilibrium potential* $g_{V,O}$, defined as the unique Green function in the above sense that moreover is harmonic on $\mathring{V} \setminus \{O\}$.

If V is embedded as a domain with C^∞ boundary in some Riemann surface (without boundary) V^+, we may extend $g_{V,O}$ by 0 on $V^+ \setminus V$. The extended function $g_{V,O}$ is continuous on $V^+ \setminus \{O\}$ and satisfies the following equality of currents on V^+:

$$\frac{i}{\pi}\partial\overline{\partial}g_{V,O} = \delta_O - \mu_{V,O},$$

where $\mu_{V,O}$ denotes a probability measure supported by the boundary ∂V of V, the so-called *harmonic measure* associated to the point O in V, which is actually defined by a positive C^∞ density on the smooth compact curve ∂V.

Moreover, by means of $g_{V,O}$ we may equip the tangent line $T_O V$ with a canonical norm, the *capacitary norm* $\|.\|^{\mathrm{cap}}_{V,O}$, which may be defined as follows, in terms of a local analytic coordinate z near O and of the function h in condition (0.3.5):

$$\|(\partial/\partial z)_{|P}\|^{\mathrm{cap}}_{V,O} = e^{-h(P)}. \tag{0.3.8}$$

The normal bundle of P in $\widehat{\mathcal{V}}$, $N_P\widehat{\mathcal{V}}$, is a line bundle over the section P. Its pull-back $P^*N_P\widehat{\mathcal{V}}$ defines a line bundle over $\operatorname{Spec}\mathbb{Z}$. The differential of $\iota : \widehat{\mathcal{V}}_{\mathbb{C}} \xrightarrow{\sim} \widehat{V}_O$ induces an isomorphism:

$$(P^*N_P\widehat{\mathcal{V}})_{\mathbb{C}} \xrightarrow[\sim]{D\iota_{|P_{\mathbb{C}}}} T_O V.$$

Using this isomorphism, the complex line $(P^*N_P\widehat{\mathcal{V}})_{\mathbb{C}}$ may be equipped with the capacitary norm $\|.\|^{\mathrm{cap}}_{V,O}$, and we may attach to $\widetilde{\mathcal{V}}$ the following Hermitian line bundle over $\operatorname{Spec}\mathbb{Z}$:

$$\overline{N}_P\widetilde{\mathcal{V}} := (P^*N_P\widehat{\mathcal{V}}, \|.\|^{\mathrm{cap}}_{V,O}).$$

Its Arakelov degree $\widehat{\deg}\,\overline{N}_P\widetilde{\mathcal{V}}$ turns out to be a fundamental invariant of $\widetilde{\mathcal{V}}$. It may also be interpreted as the self-intersection of the Arakelov divisor $(P, g_{V,O})$:

$$\widehat{\deg}\,\overline{N}_P\widetilde{\mathcal{V}} = (P, g_{V,O}) \cdot (P, g_{V,O}). \tag{0.3.9}$$

As in [Bos20, Chapter 10], we shall say that $\widetilde{\mathcal{V}}$ is *pseudoconcave* when the following positivity condition is satisfied:

$$\widehat{\deg}\,\overline{N}_P\widetilde{\mathcal{V}} > 0. \tag{0.3.10}$$

To a large extent, this memoir is an exploration of the consequences of this pseudoconcavity condition concerning the morphisms from $\widetilde{\mathcal{V}}$ to arithmetic schemes, and in particular to arithmetic surfaces.

0.3.4

Among the smooth formal-analytic surfaces $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, (V, O), \iota)$ over $\operatorname{Spec}\mathbb{Z}$, the ones such that the Riemann surface V is simply connected admit a simple description. Up to isomorphism, these are the formal-analytic surfaces $\widetilde{\mathcal{V}}(\overline{D}(0;1), \psi)$ associated to some formal series ψ in $\mathbb{R}[[X]]$ such that:

$$\psi(0) = 0 \quad \text{and} \quad \psi'(0) \neq 0, \tag{0.3.11}$$

by means of the following construction.

By definition, $\widetilde{\mathcal{V}}(\overline{D}(0;1),\psi)$ is the formal-analytic arithmetic surface $(\widehat{\mathcal{V}},(V,O),\iota)$, where:[11]

$$\widehat{\mathcal{V}} := \operatorname{Spf}\mathbb{Z}[[X]], \quad V := \overline{D}(0;1), \quad O = 0,$$

and:

$$\iota := \psi : \widehat{\mathcal{V}}_{\mathbb{C}} \simeq \operatorname{Spf}\mathbb{C}[[X]] \xrightarrow{\sim} \operatorname{Spf}\mathbb{C}[[z]] \simeq \widehat{D(0;1)}_0.$$

These formal-analytic arithmetic surfaces appear implicitly in [CDT25], through the associated algebra of regular functions $\mathcal{O}(\widetilde{\mathcal{V}}(\overline{D}(0;1),\psi))$. This algebra admits an elementary description, as the ring of formal series $\widehat{\alpha}$ in $\mathbb{Z}[[T]]$ such that the complex formal series $\widehat{\alpha}\circ\psi^{-1}$ in $\mathbb{C}[[z]]$—where ψ^{-1} denotes the compositional inverse of ψ—is the Taylor expansion at 0 of some function α^{an} holomorphic on some open neighborhood of $\overline{D}(0;1)$ in $\mathbb{C}$, or equivalently has a radius of convergence > 1.[12]

The equilibrium potential $g_{V,O}$ attached to $(V,O) := (\overline{D}(0;1),0)$ is the function $(z \mapsto \log^+ |z|^{-1})$, and the harmonic measure $\mu_{V,O}$ is the rotation-invariant probability measure on the circle $\partial\overline{D}(0;1)$.

The Hermitian line bundle $N_{\mathcal{P}}\widetilde{\mathcal{V}}(\overline{D}(0;1),\psi)$ may be identified with $(\mathbb{Z}\,\partial/\partial X, \|.\|_{\psi})$ where the metric $\|.\|_{\psi}$ satisfies:

$$\|\psi'(0)^{-1}\partial/\partial X\|_{\psi} = 1.$$

Consequently,

$$\widehat{\deg}\, N_P\widetilde{\mathcal{V}}(\overline{D}(0;1),\psi) = \log|\psi'(0)|^{-1},$$

and the pseudoconcavity condition (0.3.10) is satisfied if and only if:

$$|\psi'(0)| < 1.$$

0.3.5

In this subsection, we present a first result concerning the geometry of pseudoconcave formal-analytic arithmetic surfaces, in which the notions of arithmetic intersection theory introduced in 0.3.2 and 0.3.3 naturally enter.

0.3.5.1

Consider a smooth formal-analytic formal surface $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}},(V,O),\iota)$ over $\operatorname{Spec}\mathbb{Z}$ as above, and assume that V is equipped with a C^∞ volume form invariant under complex conjugation.

[11] Recall that $\overline{D}(0;1)$ denotes the closed unit disk of center 0 and radius 1 in $\mathbb{C}$.

[12] In [CDT25], the authors typically work with $\varphi := \psi^{-1}$ directly, so that the pseudoconcavity condition below reads $|\varphi'(0)| > 1$. We refer the reader to 8.3.2 below for a precise discussion on how to interpret the arithmetic holonomicity theorem of [CDT25] in the setting of formal-analytic arithmetic surfaces.

Let:

$$\overline{E} := (\widehat{E}, E^{\mathrm{an}}, \varphi, \|.\|)$$

be a Hermitian vector bundle over $\widetilde{\mathcal{V}}$.

To these data, we may attach the topological $\mathbb{Z}$-module $\Gamma(\widehat{\mathcal{V}}, \widehat{E})$ of global sections of $\widehat{E}$ over $\widehat{\mathcal{V}}$—it is a finitely generated projective $\mathbb{Z}[[X]]$-module—and the space $\Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|)$ of complex analytic sections s of E over $\mathring{V}$ such that:

$$\|s\|^2_{L^2} := \int_{\mathring{V}} \|s(x)\|^2 d\mu(x)$$

is finite.

Endowed with the norm $\|.\|_{L^2}$, the space $\Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|)$ is a complex Hilbert space, equipped with a canonical real structure. Moreover, the topological $\mathbb{Z}$-module $\Gamma(\widehat{\mathcal{V}}, \widehat{E})$ and the Hilbert space $(\Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|), \|.\|_{L^2})$ may be related by means of the "gluing" isomorphisms ι and φ that define $\widetilde{\mathcal{V}}$ and $\overline{E}$, respectively.

Indeed, the completed tensor product $\Gamma(\widehat{\mathcal{V}}, \widehat{E})\hat{\otimes}_{\mathbb{Z}}\mathbb{C}$ may be identified with the space $\Gamma(\widehat{\mathcal{V}}_{\mathbb{C}}, \widehat{E}_{\mathbb{C}})$ of sections of the vector bundle $\widehat{E}_{\mathbb{C}}$ over the complex formal curve $\widehat{\mathcal{V}}_{\mathbb{C}}$. In turn, the isomorphisms ι and φ determine a canonical isomorphism:

$$\Gamma(\widehat{\mathcal{V}}_{\mathbb{C}}, \widehat{E}_{\mathbb{C}}) \xrightarrow{\sim} \Gamma(\widehat{V}_O, E^{\mathrm{an}}_{\widehat{V}_O}).$$

Finally, by assigning its formal germ at O to any L^2 holomorphic section of E^{an} over $\mathring{V}$, we define a "jet map:"

$$\widehat{\eta} : \Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|) \longrightarrow \Gamma(\widehat{V}_O, E^{\mathrm{an}}_{\widehat{V}_O}) \simeq \Gamma(\widehat{\mathcal{V}}, \widehat{E})\hat{\otimes}_{\mathbb{Z}}\mathbb{C}.$$

This map is easily seen to be injective, and to be continuous when $\Gamma(\widehat{V}_O, E^{\mathrm{an}}_{\widehat{V}_O})$ is equipped with its natural Fréchet topology. The image of $\widehat{\eta}$ is actually dense in $\Gamma(\widehat{V}_O, E^{\mathrm{an}}_{\widehat{V}_O})$, and $\widehat{\eta}$ is compatible with complex conjugation.

Accordingly the triple:

$$\pi^{L^2}_{(\widetilde{\mathcal{V}},\mu)*}\overline{E} := (\Gamma(\widehat{\mathcal{V}}, \widehat{E}), (\Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|), \|.\|_{L^2}), \widehat{\eta}), \tag{0.3.12}$$

consisting of the topological $\mathbb{Z}$-module $\Gamma(\widehat{\mathcal{V}}, \widehat{E})$, the Hilbert space $(\Gamma_{L^2}(V, \mu; E^{\mathrm{an}}, \|.\|), \|.\|_{L^2})$, and the jet map $\widehat{\eta}$ is an instance of a *pro-Hermitian vector bundle over* Spec $\mathbb{Z}$ as defined in [Bos20, Chapter 5].

The pro-Hermitian vector bundles over Spec $\mathbb{Z}$ constitute an infinite-dimensional generalization of the Hermitian vector bundles over Spec $\mathbb{Z}$—that is, of Euclidean lattices—and the monograph [Bos20] develops a theory of the θ-invariants h^0_θ attached to these objects. These invariants take values in $[0, +\infty]$, and play the role, in an arithmetic setting, of the dimension over a base field k of the space of global sections $\Gamma(C, \widehat{E})$ of suitable "pro-vector bundles" $\widehat{E}$ over a projective curve C over k. In [Bos20, Chapter 7] is constructed a natural class of pro-Hermitian vector bundles $\overline{\widehat{E}}$ whose θ-invariant $h^0_\theta(\overline{\widehat{E}})$ is well-defined and finite after any "scaling" of their Hermitian structure, the *θ-finite pro-Hermitian vector bundles over* Spec $\mathbb{Z}$.

0.3.5.2

Using the definition (0.3.12): of $\pi^{L^2}_{(\widetilde{\mathcal{V}},\mu)*}\overline{E}$ and the notion of θ-finite pro-Hermitian vector bundle over $\operatorname{Spec}\mathbb{Z}$ recalled above, we may formulate the following more precise version of a basic result on pseudoconcave formal-analytic arithmetic surfaces established in [Bos20, Chapter 10]:

Theorem 0.3.1. *Let $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, (V, P), \iota)$ be a pseudoconcave smooth formal-analytic over $\mathbb{Z}$, and let μ be a C^∞ positive volume form on V invariant under complex conjugation.*

(1) For every Hermitian vector bundle $\overline{E}$ over $\widetilde{\mathcal{V}}$, the pro-Hermitian vector bundle $\pi^{L^2}_{(\widetilde{\mathcal{V}},\mu)*}\overline{E}$ is θ-finite, and we may therefore define:

$$h^0_{\theta,L^2}(\widetilde{\mathcal{V}},\mu;\overline{E}) := h^0_\theta\big(\pi^{L^2}_{(\widetilde{\mathcal{V}},\mu)*}\overline{E}\big).$$

(2) For every Hermitian line bundle $\overline{M}$ on $\widetilde{\mathcal{V}}$, when $D \in \mathbb{N}$ goes to infinity, we have:

$$h^0_{\theta,L^2}(\widetilde{\mathcal{V}},\mu;\overline{M}^{\otimes D}) = O(D^2). \tag{0.3.13}$$

More precisely, when $\overline{M}\cdot(P, g_{\widetilde{\mathcal{V}}}) < 0$, we have:

$$\lim_{D\to+\infty} h^0_{\theta,L^2}(\widetilde{\mathcal{V}},\mu;\overline{M}^{\otimes D}) = 0, \tag{0.3.14}$$

and in general:

$$\limsup_{D\to+\infty} D^{-2}\, h^0_{\theta,L^2}(\widetilde{\mathcal{V}},\mu;\overline{M}^{\otimes D}) \leq \frac{1}{2}\frac{\big(\overline{M}\cdot(P, g_{\widetilde{\mathcal{V}}_\mathbb{C}})\big)^2}{\widehat{\deg}\,\overline{N}_P\widetilde{\mathcal{V}}}. \tag{0.3.15}$$

0.3.5.3

As already indicated in [Bos20, Section 10.2], and explained in more detail in Section 1.4 of this memoir, the θ-finiteness of $\pi^{L^2}_{(\widetilde{\mathcal{V}},\mu)*}\overline{E}$ and the asymptotic estimate (0.3.13) may be seen as arithmetic analogues of some classical results of Andreotti [And63] concerning pseudoconcave complex analytic spaces.

As shown in [And63, § 3–4] for pseudoconcave complex analytic spaces, the asymptotic estimate (0.3.13) implies an algebraicity result concerning the image of morphisms from pseudoconcave formal-analytic arithmetic surfaces to arithmetic schemes:

Corollary 0.3.2 (Compare [Bos20, Theorem 10.8.1])**.** *For every pseudoconcave smooth formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ over* $\operatorname{Spec}\mathbb{Z}$ *and for every morphism:*

$$\alpha : \widetilde{\mathcal{V}} \longrightarrow X$$

from $\widetilde{\mathcal{V}}$ to some quasi-projective arithmetic scheme X, there exists a quasi-projective arithmetic surface S and a closed embedding $i : S \hookrightarrow X$ such that α factors through i.

The more precise upper-bound (0.3.15) will allow us to establish the degree bound (0.3.20) in Corollary 0.3.5 and consequently the bounds (0.3.22) and (0.3.23) in our main finiteness results, Theorems 0.3.4, 0.3.6, and 0.3.8.

0.3.6

Consider a morphism:

$$\alpha := (\widehat{\alpha}, \alpha^{\mathrm{an}}) : \widetilde{\mathcal{V}} \longrightarrow X,$$

from the smooth formal-analytic surface $\widetilde{\mathcal{V}}$ over $\operatorname{Spec} \mathbb{Z}$ with values in some normal quasi-projective arithmetic surface X, and assume that the morphism:

$$\widehat{\alpha}_{\mathbb{Q}} : \widetilde{\mathcal{V}}_{\mathbb{Q}} \longrightarrow X_{\mathbb{Q}},$$

from the germ of formal curve $\widetilde{\mathcal{V}}_{\mathbb{Q}} \simeq \operatorname{Spf} \mathbb{Q}[[T]]$ to the smooth curve $X_{\mathbb{Q}}$ over $\mathbb{Q}$, is not constant.

We may introduce the Arakelov divisor with compact support on X defined as the direct image by α of the Arakelov divisor $(P, g_{V,O})$ on $\widetilde{\mathcal{V}}$, namely:

$$\alpha_*(P, g_{V,O}) := (\widehat{\alpha}_* P, \alpha_*^{\mathrm{an}} g_{V,O}). \tag{0.3.16}$$

Its self-intersection:

$$\alpha_*(P, g_{V,O}) \cdot \alpha_*(P, g_{V,O})$$

—which makes sense in the formalism of arithmetic intersection on quasi-projective arithmetic surfaces developed in Part II—is a well-defined real number.[13]

When moreover the formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ satisfies the pseudoconcavity condition (0.3.10), the self-intersection (0.3.16) is positive, and we may attach to the morphism α the following positive invariant, which plays a central role in this monograph:

$$D(\alpha : \widetilde{\mathcal{V}} \to X) := \frac{\alpha_*(P, g_{V,O}) \cdot \alpha_*(P, g_{V,O})}{\widehat{\deg}\, \overline{N}_P \widetilde{\mathcal{V}}}. \tag{0.3.17}$$

According to (0.3.9), the invariant $D(\alpha : \widetilde{\mathcal{V}} \to X)$ is the quotient by the self-intersection of the Arakelov divisor $(P, g_{V,O})$ of the self-intersection of its direct image by α, and is therefore a natural invariant from a formal perspective. Remarkably enough, it is possible to express it in terms of classical quantities involving the "finite" and "Archimedean" components $\widehat{\alpha}$ and α^{an} of the morphism α.

[13]Observe that the direct image $\alpha_*^{\mathrm{an}} g_{V,O}$ is not a Green function with C^∞ regularity as used in classical arithmetic intersection theory, which is therefore not adequate to define this self-intersection.

For simplicity, we will only indicate here that this expression for $D(\alpha : \widetilde{\mathcal{V}} \to X)$ satisfies the lower bound:

$$D(\alpha : \widetilde{\mathcal{V}} \to X) \geq e(\alpha),$$

where $e(\alpha)$ denotes the ramification index of $\widehat{\alpha}_{\mathbb{Q}}$, and write down its special form when $\widetilde{\mathcal{V}}$ is the formal-arithmetic surface $\widetilde{\mathcal{V}}(\overline{D}(0;1), \psi)$ attached to a formal series $\psi \in \mathbb{R}[[X]]$ as in 0.3.4 and when X is the affine line $\mathbb{A}^1_{\mathbb{Z}}$:

Proposition 0.3.3. *For every* $\psi \in \mathbb{R}[[X]]$ *satisfying conditions* (0.3.11) *and every morphism:*

$$\alpha : \widetilde{\mathcal{V}}(\overline{D}(0,1), \psi) \longrightarrow \mathbb{A}^1_{\mathbb{Z}}$$

such that $\widehat{\alpha}_{\mathbb{Q}}$ *is non-constant, the following equality holds:*

$$\alpha_*(P, g_{\widetilde{\mathcal{V}}_{\mathbb{C}}}) \cdot \alpha_*(P, g_{\widetilde{\mathcal{V}}_{\mathbb{C}}}) = 2 \int_0^1 \int_0^1 \log \left| \alpha^{\mathrm{an}}(e^{2\pi i t_1}) - \alpha^{\mathrm{an}}(e^{2\pi i t_2}) \right| dt_1 \, dt_2.$$

In particular, when moreover $\widetilde{\mathcal{V}}(\overline{D}(0,1), \psi)$ *is pseudoconcave,*[14] *we have:*

$$D(\alpha : \widetilde{\mathcal{V}} \to \mathbb{A}^1_{\mathbb{Z}}) = 2 \big(\log |\psi'(0)|^{-1} \big)^{-1} \int_0^1 \int_0^1 \log \left| \alpha^{\mathrm{an}}(e^{2\pi i t_1}) - \alpha^{\mathrm{an}}(e^{2\pi i t_2}) \right| dt_1 \, dt_2. \tag{0.3.18}$$

0.3.7

Having the invariant $D(\alpha : \widetilde{\mathcal{V}} \to X)$ at our disposal, we may formulate, in a simplified setting, some of the main results of this memoir.

Theorem 0.3.4. *Let* $\widetilde{\mathcal{V}}$ *be a pseudoconcave formal-analytic arithmetic surface over* $\operatorname{Spec} \mathbb{Z}$*, and let* U *and* V *be two integral normal arithmetic surfaces. Consider a commutative diagram:*

$$\begin{array}{ccc} & & V \\ & \overset{\beta}{\nearrow} & \downarrow f \\ \widetilde{\mathcal{V}} & \xrightarrow{\alpha} & U, \end{array} \tag{0.3.19}$$

where α *and* β *are morphisms from the formal-analytic arithmetic surface* $\widetilde{\mathcal{V}}$ *to the arithmetic surfaces* U *and* V*, and where* f *is a morphism of schemes.*

If $\alpha_{\mathbb{C}}$ *is non-constant, and therefore so is* $\beta_{\mathbb{C}}$*, then* f *is dominant and generically finite, and its degree* $\deg f$ *satisfies the upper bound:*

$$\deg f \leq \frac{D(\alpha : \widetilde{\mathcal{V}} \to U)}{D(\beta : \widetilde{\mathcal{V}} \to V)}. \tag{0.3.20}$$

[14]That is, when $|\psi'(0)| < 1$.

The commutativity of the diagram (0.3.19) means, by definition, that the following two diagrams are commutative:

$$\begin{array}{ccc} & & V \\ & \overset{\widehat{\beta}}{\nearrow} & \big\downarrow f \\ \widehat{\mathcal{V}} & \xrightarrow{\widehat{\alpha}} & U, \end{array}$$

and

$$\begin{array}{ccc} & & V(\mathbb{C}) \\ & \overset{\beta^{\mathrm{an}}}{\nearrow} & \big\downarrow f_{\mathbb{C}} \\ V & \xrightarrow{\alpha^{\mathrm{an}}} & U(\mathbb{C}). \end{array}$$

Actually, the commutativity of any of these two diagrams implies the commutativity of the other one.

Observe also that, combined with the estimates:

$$D(\beta : \widetilde{\mathcal{V}} \to V) \geq e(\beta) \geq 1,$$

the degree bound (0.3.20) implies the following one:

Corollary 0.3.5. *With the notation of Theorem 0.3.4, the following inequality holds:*

$$\deg f \leq D(\alpha : \widetilde{\mathcal{V}} \to U). \tag{0.3.21}$$

The fact that the right-hand side of (0.3.21) does not depend of V or β plays a key role in the proof of diverse results concerning pseudoconcave arithmetic surfaces, for instance, Theorems 0.3.6, 0.3.7, and 0.3.8.

Theorem 0.3.6. *For every pseudoconcave smooth formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ over* Spec $\mathbb{Z}$*, the field of meromorphic functions $\mathcal{M}(\widetilde{\mathcal{V}})$ is either $\mathbb{Q}$, or a finitely generated extension of $\mathbb{Q}$ of transcendence degree* 1.

Moreover, if f is an element of $\mathcal{O}(\widetilde{\mathcal{V}})$ not in $\mathbb{Q}$, then f seen as an element of $\mathcal{M}(\widetilde{\mathcal{V}})$ is transcendental over $\mathbb{Q}$, and the degree of $\mathcal{M}(\widetilde{\mathcal{V}})$ as a field extension of $\mathbb{Q}(f)$ satisfies the following upper bound:

$$[\mathcal{M}(\widetilde{\mathcal{V}}) : \mathbb{Q}(f)] \leq D(f : \widetilde{\mathcal{V}} \to \mathbb{A}^1_{\mathbb{Z}}). \tag{0.3.22}$$

In the main body of the text, we establish more general forms of Theorems 0.3.4 and 0.3.6, where $\widetilde{\mathcal{V}}$ may be a smooth formal-analytic arithmetic surface over Spec $\mathcal{O}_K$, with K an arbitrary number field and $\mathcal{O}_K$ its ring of integers, and where the maps α, β, or f are allowed to be, not only morphisms, but more general *meromorphic maps*.

In particular, the second half of Theorem 0.3.6 still holds when f is an arbitrary element of $\mathcal{M}(\widetilde{\mathcal{V}})$ not in $\mathbb{Q}$, with a suitable definition of the invariant $D(f)$ on the right-hand side of (0.3.22).

0.3.8

By elaborating on the degree bounds established in Theorems 0.3.4 and 0.3.6, it is possible to obtain further results concerning pseudoconcave formal-analytic arithmetic surfaces and their morphisms to quasi-projective arithmetic surfaces. We conclude this section by presenting two of these results.

0.3.8.1

For every smooth formal-analytic surface $\widetilde{\mathcal{V}}$ over $\operatorname{Spec}\mathbb{Z}$, the $\mathbb{Z}$-algebra $\mathcal{O}(\widetilde{\mathcal{V}})$ is a domain. Its fraction field $\operatorname{Frac}\mathcal{O}(\widetilde{\mathcal{V}})$ may be identified to a subfield of $\mathcal{M}(\widetilde{\mathcal{V}})$, but in general may be distinct from $\mathcal{M}(\widetilde{\mathcal{V}})$.

According to Theorem 0.3.6, this fraction field is a finitely generated extension of $\mathbb{Q}$, of transcendence degree at most 1. The following theorem establishes a stronger finiteness result:

Theorem 0.3.7. *For every pseudoconcave smooth formal-analytic arithmetic surface $\widetilde{\mathcal{V}}$ over* $\operatorname{Spec}\mathbb{Z}$, *the $\mathbb{Z}$-algebra $\mathcal{O}(\widetilde{\mathcal{V}})$ is finitely generated.*

0.3.8.2

As in 0.3.6 above, let:

$$\alpha := (\widehat{\alpha}, \alpha^{\mathrm{an}}) : \widetilde{\mathcal{V}} \longrightarrow X$$

be a morphism from a smooth formal-analytic surface $\widetilde{\mathcal{V}} := (\widehat{\mathcal{V}}, (V, O), \iota)$ over $\operatorname{Spec}\mathbb{Z}$ with value in some normal quasi-projective arithmetic surface X, and assume that the morphism $\widehat{\alpha}_{\mathbb{Q}}$ is non-constant.

We may consider the étale fundamental groups $\pi_1^{\text{ét}}(V, O)$ and $\pi_1^{\text{ét}}(X, \alpha^{\mathrm{an}}(O))$. The former may be identified with the profinite completion of the topological fundamental group $\pi_1(V, O)$, and the map:

$$V \xrightarrow{\alpha^{\mathrm{an}}} X_{\mathbb{C}} \longrightarrow X$$

defines a continuous morphism of profinite groups:

$$\alpha_* : \pi_1^{\text{ét}}(V, O) \longrightarrow \pi_1^{\text{ét}}(X, \alpha^{\mathrm{an}}(O)).$$

The following theorem is an arithmetic avatar of a generalization due to Nori [Nor83] of the classical theorem of Lefschetz concerning the fundamental groups of hyperplane sections of projective complex varieties.

Theorem 0.3.8. *With the above notation, if $\widetilde{\mathcal{V}}$ is pseudoconcave, then $\alpha_*(\pi_1^{\text{ét}}(V, O))$ is a subgroup of finite index in $\pi_1^{\text{ét}}(X, \alpha^{\mathrm{an}}(O))$. Moreover,*

$$[\pi_1^{\text{ét}}(X, \alpha^{\mathrm{an}}(O)) : \alpha_*(\pi_1^{\text{ét}}(V, O))] \leq D(\alpha : \widetilde{\mathcal{V}} \to X). \qquad (0.3.23)$$

(continued...)

Notation index

Index