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Chapter 1

Introduction

In this book, we prove the standard endoscopic fundamental lemma for the spherical Hecke algebra of an unramified reductive group and its unramified endoscopic groups. A more precise version of the following result may be found in Theorem 2.6.17.

Theorem 1.0.1. *Let $k = \mathbb{F}_q$ be a finite field with q elements and characteristic p, $\mathcal{O} = k[[\pi]]$ the ring of power series of one variable over k, and $F = k((\pi))$ its field of fractions. Let G be a reductive group scheme over $\mathcal{O}$ whose Coxeter number is less than $p/2$. Let $(\kappa, \vartheta_\kappa, \xi)$ be an endoscopic datum of G over $\mathcal{O}$ and H the corresponding endoscopic group. Then we have equality of orbital integrals*

$$\Delta_0(\gamma_H, \gamma)\,\mathbf{O}_a^\kappa\big(f^V, \mathrm{d}t\big) = \mathbf{SO}_{a_H}\big(f^V_{H,\xi}, \mathrm{d}t\big),$$

where a and a_H are matching strongly regular semisimple conjugacy classes represented by $\gamma \in G(F)$ and $\gamma_H \in H(F)$ respectively, f^V is the Satake function associated with an ${}^LG$-representation V and $f^V_{H,\xi}$ is the corresponding Satake function of H by transferring through L-embedding ξ, and $\Delta_0(\gamma_H, \gamma)$ is the (appropriately defined version of the) transfer factor.

The Lie algebra analogue of Theorem 1.0.1 was proved by B.C. Ngô in his monumental work [Ngô10]. Based on Ngô's result, people were able to deduce several analogous results. First, when it is combined with the work of Waldspurger [Wal06], one can deduce the Lie algebra fundamental lemma over local number fields. Another work by Waldspurger [Wal97] then allows us to deduce the analogue of Theorem 1.0.1 for the unit element of the spherical Hecke algebra over local number fields. Later, Hales [Hal95] used some simple trace formula to further obtain the result for the entire spherical Hecke algebra, again over local number fields. Finally, Casselman, Cely, and Hales in [CCH19] used motivic integration to deduce a slightly weaker version of Theorem 1.0.1 from [Hal95], where the characteristic of k is assumed to be "large" rather than have an explicit lower bound.

For many applications in number theory, the historical results recounted above may be viewed as "complete." However, there are still deficiencies from a development point of view. For example, the chain of logic across several similar yet different situations, namely, number fields and function fields, and Lie algebras and groups, seems unnecessarily convoluted. It would already be very interesting if a direct proof could be found for the group case. For another example, the proof of [Ngô10] is entirely geometric and provides great insights for understanding the trace formula for Lie algebras through a geometric lens, almost all of which are not reflected in later works mentioned above. Moreover, the trace formula for Lie algebra is only a watered-down version of the bona fide trace formula, and many interesting phenomena are lost. Therefore, a direct, geometric proof of Theorem 1.0.1 in the style of [Ngô10] is very much desirable.

For these reasons, the organizational structure of [Ngô10] is used as a base template for this book, and comparison with the Lie algebra case will be a recurring theme. The central object of this book is the multiplicative analogue of Hitchin fibration in the case of the adjoint action of a reductive group. We will call it the *multiplicative Hitchin fibration*, or *mH-fibration* for short. Such generalization in arithmetic setting was first envisioned in [FN11], and some important preliminary explorations were done notably in [Bou15, Bou17, BC18, Chi22].

In recent years, the so-called relative Langlands program has been of great interest, where a much more general class of trace formulae involving spherical varieties are considered. Therefore, the current book also serves as a proof of concept for future development. To this end, in this introduction we will try to give an overview of this book with emphases on the big picture, on new features, and on potential development points.

The geometry of mH-fibration is incredibly rich and beautiful and has a profound connection with representation theory; therefore it is important to give enough spotlight to all its key features in this introduction so that the reader can have a grasp on what is to come. However, to keep the introduction concise and clear enough, we will try to avoid using too many notations due to the technical complexity of the subject.

1.1 ON HITCHIN-TYPE FIBRATIONS

One of the motivations behind introducing Hitchin fibration to arithmetic problems like the fundamental lemma is that global objects tend to behave much better than local ones, even if they are less computationally accessible at times. Therefore, it would be beneficial to understand the general principles of these Hitchin-type global constructions. We will use mH-fibrations as the primary example for this section.

1.1.1. There have been many attempts to prove the fundamental lemma for reductive groups or Lie algebras in the past by focusing on the local picture. Over Archimedean local fields, Shelstad was able to prove the general smooth transfer directly at group level (see [She79]), thus bypassing the need for a fundamental lemma. Over non-Archimedean fields, however, the problem seems much harder, and most proofs before [Ngô10] were on one specific group at a time. The reader can see the introduction of [Ngô10] for more historical details.

One of the most successful general results on the local front is perhaps the conditional proof for unramified conjugacy classes by Goresky, Kottwitz, and MacPherson in [GKM04]. Although quite an impressive framework in itself, it also shows the limitation of the local geometric method. For one, as Ngô pointed out in [Ngô10], it depends on a purity conjecture of the cohomology of related affine Springer fibers, which is only partially known due to affine Springer fibers usually being highly singular. Another serious obstacle pointed out by [Ngô10] is that the proof crucially depends on the conjugacy class's being unramified so that there is a large torus acting on the affine Springer fibers.

1.1.2. Roughly speaking, the local geometric method is based on the fact that orbital integrals may be interpreted as certain kinds of point-counting on related affine Springer fibers, which in turn reduces the fundamental lemma to some cohomological statement using a variant of the Grothendieck–Lefschetz trace formula. The affine Springer fiber in question can be roughly understood as a certain subset of maps from a formal disc $X_v = \operatorname{Spec} \mathcal{O}$ to the quotient stack $[\mathfrak{g}/G]$, together with some other naturally attached data. To move from the local picture to the global one, one only needs to replace the formal disc by a smooth projective curve X. Because $\mathfrak{g}$ is affine and X is projective, one needs to add some auxiliary twist; otherwise the global object we end up getting will be trivial. In the Lie algebra case, such a twist is provided by the natural $\mathbb{G}_{\mathrm{m}}$-action on $\mathfrak{g}$ viewed as a vector space.

More formally, the classical Hitchin fibration can be formulated as follows: by the Chevallay restriction theorem, the GIT quotient $\mathfrak{c} = \mathfrak{g} /\!\!/ G \simeq \mathfrak{t} /\!\!/ W$ is an affine space. The $\mathbb{G}_{\mathrm{m}}$-scaling on $\mathfrak{g}$ commutes with the adjoint action, so the map $\mathfrak{g} \to \mathfrak{c}$ is $\mathbb{G}_{\mathrm{m}}$-equivariant. Consider the two-stage map

$$[\mathfrak{g}/G \times \mathbb{G}_{\mathrm{m}}] \longrightarrow [\mathfrak{c}/\mathbb{G}_{\mathrm{m}}] \longrightarrow \mathbb{B}\mathbb{G}_{\mathrm{m}},$$

where $\mathbb{B}\mathbb{G}_{\mathrm{m}}$ is the classifying stack of $\mathbb{G}_{\mathrm{m}}$. Applying the mapping stack functor from X to the first map, we obtain the *universal Hitchin fibration* associated with $\mathfrak{g}$ and X, lying over the Picard stack of X. Taking the fiber over the canonical sheaf of X, or, more generally, any line bundle, we obtain the Hitchin fibration in the usual sense.

1.1.3. Without going further into the global geometry, we must first ask what the analogue is for the above setup in the group case. Clearly we are now interested in the quotient stack $[G/G]$ (where G acts on itself by adjoint action) in place of $[\mathfrak{g}/G]$, but we also need a natural twisting similar to the $\mathbb{G}_{\mathrm{m}}$-action on $\mathfrak{g}$. If $G = \mathrm{GL}_n$, for example, we have the scaling action of $\mathbb{G}_{\mathrm{m}}$ by viewing G as a subspace of Mat_n, and for a group G with a nontrivial central torus Z_0, we may use the action of Z_0.

There are several problems using Z_0, however. For one, this action heavily depends on the situation, and does not exist if G is semisimple. Even if Z_0 is not trivial, it still does not provide enough twisting if it is anisotropic. There is also another difficulty: even for $G = \mathrm{GL}_1 = \mathbb{G}_{\mathrm{m}}$, if we twist G by a $\mathbb{G}_{\mathrm{m}}$-torsor $\mathcal{L}$, then the induced bundle $G_{\mathcal{L}}$ is just $\mathcal{L}$ itself, which has no global sections unless $\mathcal{L}$ is trivial. Compare this to the Lie algebra case when $\mathfrak{g} = \mathbb{A}^1$: $\mathfrak{g}_{\mathcal{L}}$ is the line bundle associated with the $\mathbb{G}_{\mathrm{m}}$-torsor $\mathcal{L}$, which does have a lot of sections provided that the degree is large enough.

1.1.4. There is a relatively straightforward way to solve this lack-of-twist problem: since there is no nonconstant map from X to $[G/G]$, we may instead consider rational maps. This is the same idea as that behind global Hecke stacks, and in order to obtain a reasonable geometric object, we also need a systematic way to control the poles of the rational maps, similarly to truncated Hecke stacks. This is in fact the formulation used by some prototypes of mH-fibrations in the field of mathematical physics (see, for example, [HM02]), as well as in its first appearance in arithmetic settings, in [FN11].

One downside (among many others) of this "Hecke-style" formulation is that some technical aspects become very messy and unwieldy if $G^{\mathrm{sc}} \neq G^{\mathrm{ad}}$ or G is not split. On the other hand, the mapping stack approach used in the Lie algebra case is very clean when dealing with non-split groups, so pursuing that route seems more promising.

Since we cannot guarantee that G contains a central torus, we can certainly add one to it by augmenting the group. The question is of course what torus to add. Moreover, the $G = \mathbb{G}_{\mathrm{m}}$ versus $\mathfrak{g} = \mathbb{A}^1$ example above shows we need to add some "boundaries" to G so that the mapping stack contains more than just constant maps. This is where reductive monoids enter the picture.

1.1.5. A primal example of a reductive monoid is that of $n \times n$ matrices Mat_n. The group GL_n embeds in Mat_n as an open subset and is its group of units. Therefore, $\mathrm{Mat}_n - \mathrm{GL}_n$ is the "boundary" we could add to GL_n. Coincidentally, $\mathrm{Mat}_n \cong \mathfrak{gl}_n$, so our first example of mH-fibration is the same Hitchin fibration associated with $\mathfrak{gl}_n$.

The Mat_n example is somewhat misleading, because it is essentially the only case where the monoid is smooth. In general, the Lie algebra is replaced by a *very flat* reductive monoid $\mathfrak{M}$ (see § 2.3), and the role of $\mathbb{G}_{\mathrm{m}}$ is played by the center $Z_{\mathfrak{M}}$ of the unit group $\mathfrak{M}^{\times}$. There is a maximal toric variety $\mathfrak{T}_{\mathfrak{M}}$ playing the role of Cartan subalgebra, and it is known that $\mathfrak{C}_{\mathfrak{M}} := \mathfrak{M} /\!\!/ G \simeq \mathfrak{T}_{\mathfrak{M}} /\!\!/ W$. However, there is an extra step for mH-fibrations.

1.1.6. The monoid $\mathfrak{M}$ comes with an *abelianization* map $\mathfrak{M} \to \mathfrak{A}_{\mathfrak{M}}$ introduced by Vinberg in [Vin95] by taking the GIT quotient by the $G^{\mathrm{sc}} \times G^{\mathrm{sc}}$-multiplication on the left and right. We will be primarily interested in the case where the commutative monoid $\mathfrak{A}_{\mathfrak{M}}$ is an affine space (the general case can be dealt with after some technical work but does not add anything new). It can be shown that

$$\mathfrak{C}_{\mathfrak{M}} \simeq \mathfrak{A}_{\mathfrak{M}} \times \mathfrak{C},$$

where $\mathfrak{C} := G^{\mathrm{sc}} /\!\!/ G^{\mathrm{sc}}$ is an affine space. The upshot is that we have a three-stage map

$$[\mathfrak{M}/G \times Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow [\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}] \longrightarrow \mathbb{B}Z_{\mathfrak{M}},$$

and we may apply the mapping stack functor to these maps. The resulting stacks are denoted by

$$\mathcal{M}_X^+ \longrightarrow \mathcal{A}_X^+ \longrightarrow \mathcal{B}_X^+ \longrightarrow \mathrm{Bun}_{Z_{\mathfrak{M}}}.$$

1.1.7. There is a maximal open Deligne–Mumford substack $\mathcal{B}_X \subset \mathcal{B}_X^+$ which is essentially a moduli stack of "divisors" on X with values in the dominant cocharacters of G (see §§ 5.2 and 6.1). Such divisors are usually called *colored divisors* in some literature, and are called *boundary divisors* in this book. Taking inverse images, we have stacks

$$\mathcal{M}_X \longrightarrow \mathcal{A}_X \longrightarrow \mathcal{B}_X \longrightarrow \mathrm{Bun}_{Z_{\mathfrak{M}}},$$

the first step of which is the *universal mH-fibration* associated with G, $\mathfrak{M}$, and X. Unlike the Lie algebra case where the $\mathbb{G}_\mathrm{m}$-torsor is fixed, we will *not* fix a $Z_{\mathfrak{M}}$-torsor.

In earlier papers like [Bou15, Bou17, Chi22], $\mathfrak{M}$ is restricted to the so-called *Vinberg monoid* or *universal monoid* of G^{sc} (denoted by $\mathrm{Env}(G^{\mathrm{sc}})$), while in [FN11] the monoid is what is commonly called an *L-monoid*, associated with a single fixed dominant cocharacter λ of G (denoted by $\mathfrak{M}(\lambda)$). In this book we will consider an arbitrary very flat monoid $\mathfrak{M}$, thus unifying different formulations into one framework. The main reason for doing this is not merely technical and is only apparent when we start to consider endoscopic groups.

1.1.8. We will continue discussing mH-fibrations in the next section, and close this one with a few comments on potential generalizations. Instead of limiting ourselves to just G acting on G of $\mathfrak{M}$, we may consider a G-space M and an equivariant partial compactification $\mathfrak{M}$. Suppose we also have some group $Z_{\mathfrak{M}}$ of multiplicative type acting on $\mathfrak{M}$ commuting with the G-action; then we may also consider the Hitchin-type fibration associated with $[\mathfrak{M}/G \times Z_{\mathfrak{M}}] \to [(\mathfrak{M} /\!\!/ G)/Z_{\mathfrak{M}}]$.

In [SV17] and later [BZSV24], the authors proposed a unified framework for relative trace formulae using spherical varieties. In the group case, G is viewed as a $G \times G$-spherical variety, and $\mathrm{Env}(G^{\mathrm{sc}})$ is a horospherical contraction of G^{sc} as a spherical variety and also the spectrum of the Cox ring of the wonderful compactification of G^{ad} (one may consult [Tim11] for terminologies). Therefore, a particularly promising generalization is the following: if we have a spherical homogeneous space G/H which admits a wonderful compactification $\overline{G/H}$, we may consider the spectrum $\mathfrak{M}$ of the Cox ring of $\overline{G/H}$. In this case, there is a canonical torus $Z_{\mathfrak{M}}$ acting on $\mathfrak{M}$ commuting with the action of G, and so it descends to the abelianization $\mathfrak{A}_{\mathfrak{M}} := \mathfrak{M} /\!\!/ G$ as in the monoid case. The abelianization $\mathfrak{A}_{\mathfrak{M}}$ is known to be an affine space. Thus, we may consider the Hitchin-type fibration associated with any subgroup (for example, isotropy group H itself) of G acting on $\mathfrak{M}$.

1.2 LOCAL MODEL OF SINGULARITY

After we define mH-fibrations using reductive monoids, a lot of their geometric properties can be laid out in a fashion parallel to that of [Ngô10], often without too much modification to the argument. Doing so requires detailed study of the invariant theory of the G-action on $\mathfrak{M}$, which is largely done in [Bou15, Chi22], and we only record the results and bridge some remaining gaps. One main exception to all this "house-keeping" is studying the deformation of multiplicative Higgs (mHiggs) bundles, which has not been done so far in a satisfactory way.

1.2.1. In [Ngô10, § 4.14], Ngô proved that the total stack of Hitchin fibration is smooth under the assumption that the twisting line bundle is very ample. His result is based on the tangent-obstruction theory of Higgs bundles due to Biswas and Ramanan in [BR94]. The group case is more complicated and the total stack of mH-fibration is usually not smooth. As a result, we must replace constant sheaf by the intersection complex in our later study of cohomologies. The question is, then, what do the singularities of the total stack look like? In [FN11, Conjecture 4.2] the authors conjectured that the intersection complex is exactly the globalized version of some Satake sheaf. We will prove this conjecture essentially in full in this book by establishing a tangent-obstruction theory for mHiggs bundles. Below is a rather vaguely stated corollary of Theorems 6.10.2 and 6.10.10, which proves [FN11, Conjecture 4.2]:

Theorem 1.2.2. *Let $h_X\colon \mathcal{M}_X \to \mathcal{A}_X$ be the mH-fibration associated with G, $\mathfrak{M}$, and curve X, and let $\mathcal{B}_X$ be its moduli of boundary divisors. The pullback complex* $\mathrm{ev}^*\mathrm{IC}_{[\mathsf{Q}_X]}$ *is, up to shifts and Tate twists, isomorphic to the intersection complex on $\mathcal{M}_X$ after restricting to a "very large" open subset of $\mathcal{A}_X$. Here $[\mathsf{Q}_X] \to \mathcal{B}_X$ is a certain truncated global Hecke stack induced by monoid $\mathfrak{M}$ and parametrized by boundary divisors in $\mathcal{B}_X$ (see § 5.3), and* $\mathrm{ev}\colon \mathcal{M}_X \to [\mathsf{Q}_X]$ *is an appropriately defined evaluation map over $\mathcal{B}_X$.*

We avoid explaining more notations in Theorem 1.2.2 because they are technical and will drag this introduction longer than necessary. However, an explanation of the phrase "very large" is due in order to understand the novelty in this result. In [Bou17], a weaker version of Theorem 1.2.2 was proved using an ad hoc method. The author did not try to establish a tangent-obstruction theory in that paper, and thus had to impose a much stronger technical condition that is even more difficult to state.

The gist is that Bouthier's version of "very large open subset" is not large enough and *never* includes any stratum coming from endoscopy (outside some very rare cases, essentially only when $G = \mathrm{SL}_2$). It makes the old result unusable in tackling the fundamental lemma. In our new results, the size of the open subset is on par with the analogous result in [Ngô10].

1.2.3. The most important implication of Theorem 1.2.2 is that the intersection complex on $\mathcal{M}_X$ is essentially a product of local IC-sheaves on affine Schubert varieties. More generally, if we have at each place v of X a local Satake sheaf $\mathcal{Q}_v$, we can assemble them into a pure perverse sheaf $\mathcal{Q}$ on $\mathcal{M}_X$. This is essentially a geometrization of representations of global L-groups. Once we start to consider endoscopic groups (and their mH-fibrations), it will be very clear how to transfer $\mathcal{Q}$

to endoscopic side, and we will be able to geometrize the stabilization process as well, similarly to [Ngô10, § 6.4].

1.2.4. We briefly go back to the hypothesized generalization involving a wonderful variety $\overline{G/H}$ at the end of § 1.1. Since in this case we can still define the moduli stack of boundary divisors using $[\mathfrak{A}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$, we should be able to define $[\mathsf{Q}_X]$ analogously (see § 5.3). Naturally, we expect to have a statement of geometric Satake isomorphism and a local model of singularity as well, but to do so we must first have a good framework dealing with ℓ-adic sheaves on the quotient of an arc space by an arc group, which does not seem to exist in published form at the time of writing.

1.3 GEOMETRIC TRANSFER

Utilizing universal monoid $\mathfrak{M} = \mathrm{Env}(G^{\mathrm{sc}})$ and the associated mH-fibration, A. Bouthier and J. Chi in [Bou15, BC18], and later J. Chi in [Chi22], were able to prove the dimension formula for the multiplicative affine Springer fibers of G. As we have discussed before, A. Bouthier in [Bou17] also obtained global results by studying the singularities of the total stack of mH-fibrations. However, once one attempts to connect mH-fibrations of G to those of endoscopic group H, things quickly fall apart if one only sticks with $\mathrm{Env}(G^{\mathrm{sc}})$ and $\mathrm{Env}(H^{\mathrm{sc}})$.

1.3.1. In the Lie algebra case, if H happens to be a subgroup of G, then we have the natural map $[\mathfrak{h}/H] \to [\mathfrak{g}/G]$, and since both $\mathfrak{h}$ and $\mathfrak{g}$ are reductive Lie algebras, the general results for Hitchin fibrations can be applied to both sides. In general, H is not a subgroup of G; nevertheless there is still a canonical map from $\mathfrak{c}_H = \mathfrak{h} /\!\!/ H$ to $\mathfrak{c}$, which induces a closed embedding from the Hitchin base of H to that of G. This map is all we need for establishing the geometric (spherical) transfer in the Lie algebra case.

In the group case, however, there is usually no way to directly relate $\mathrm{Env}(H^{\mathrm{sc}})$ to $\mathrm{Env}(G^{\mathrm{sc}})$ even if H is a subgroup of G, and more generally there is no map from $\mathfrak{C}_{\mathrm{Env}(H^{\mathrm{sc}})} = \mathrm{Env}(H^{\mathrm{sc}}) /\!\!/ H$ to $\mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})} = \mathrm{Env}(G^{\mathrm{sc}}) /\!\!/ G$ either. Using the general theory of (not necessarily very flat) reductive monoids, one can still produce some canonical monoid $\mathfrak{M}'_H$ for H such that $\mathfrak{M}'_H /\!\!/ H$ does map to $\mathfrak{C}_{\mathrm{Env}(G^{\mathrm{sc}})}$, and if H happens to be a subgroup of G, $\mathfrak{M}'_H$ is just the closure of the group $H' \subset \mathfrak{M}^\times$, where H' has the same semisimple type as H. However, such monoid $\mathfrak{M}'_H$ in general is not a very flat monoid, so its corresponding mH-fibration will not have very good geometric properties.

1.3.2. This, of course, is not a coincidence, and (in my opinion) is where the story becomes fascinating. The picture is better understood once we move to the dual group side, and the solution to this difficulty will manifest itself once we do so.

The fundamental lemma is usually stated as an equality between the evaluation of orbital integrals at a spherical function. In the Lie algebra case, the function is already given, namely the characteristic function on $\mathfrak{g}(\mathcal{O})$ and that on $\mathfrak{h}(\mathcal{O})$, so all we need is to figure out how to match conjugacy classes. In the group case, however, we have to geometrize both the matching of conjugacy classes and the matching of functions. The latter has close connection to the representations of L-groups thanks to geometric Satake isomorphism.

1.3.3. Temporarily going back to the local setting, we can see that the effect of the $\mathbb{G}_{\mathrm{m}}$-twisting in the Lie algebra case is that we are not really considering the characteristic function on $\mathfrak{g}(\mathcal{O})$, but rather those on scaled subsets $\pi^{-d}\mathfrak{g}(\mathcal{O}) \subset \mathfrak{g}(F)$ for arbitrary $d \geq 0$. Since these subsets do not look very different from $\mathfrak{g}(\mathcal{O})$, we are still effectively only considering a single function. In the group case, however, things get complicated because the analogue would be Cartan decomposition and Satake functions on the closure of double cosets $G(\mathcal{O})\pi^{\lambda}G(\mathcal{O})$, and these functions certainly do not look alike.

1.3.4. Recall that to each mH-fibration we also have the moduli $\mathcal{B}_X$ of boundary divisors, which are essentially divisors valued in dominant cocharacters of G. The cocharacters λ that can appear depends on the choice of monoid $\mathfrak{M}$, essentially (but not strictly) in a bijective way. Theorem 1.2.2 implies that we can encode Satake sheaves that are supported on affine Schubert variety $\mathsf{Gr}_G^{\leq\lambda}$ into $\mathcal{M}_X$. Using geometric Satake isomorphism, by carefully choosing $\mathfrak{M}$, we can encode any representation of L-group ${}^LG$ into mH-fibrations.

1.3.5. If we have an admissible embedding, otherwise known as L-embedding, of L-groups ${}^LH \to {}^LG$, we can “transfer” a representation of ${}^LG$ to ${}^LH$, namely by restriction. Since we have a (roughly bijective) correspondence between monoid $\mathfrak{M}$ and representations of L-groups, it gives hints on how to relate mH-fibrations for G to those for H.

With this consideration about the L-group in mind, we will be able to construct for any given monoid $\mathfrak{M}$ of G a canonically associated monoid $\mathfrak{M}_H$ (see Definition 2.5.21) of H such that there exists a canonical map of quotient space

$$\mathfrak{C}_{\mathfrak{M},H} = \mathfrak{M}_H /\!\!/ H \longrightarrow \mathfrak{C}_{\mathfrak{M}}.$$

1.3.6. Finding the correct monoid, however, is not the end of the story. The reason is very simple: mH-base $\mathcal{A}_X$ is constructed from stack $[\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$, but the monoids $\mathfrak{M}$ and $\mathfrak{M}_H$ have different centers, and there is usually no map from $Z_{\mathfrak{M},H}$ (the center of $\mathfrak{M}_H^{\times}$) to $Z_{\mathfrak{M}}$ whatsoever. For example, suppose $G = \mathrm{SL}_2$ and $\mathfrak{M} = \mathrm{Mat}_2$, and H is a one-dimensional torus; then $\mathfrak{M}_H = \mathbb{A}^2$ is the monoid of diagonal matrices in $\mathfrak{M}$. In this case $Z_{\mathfrak{M},H}$ is the group of invertible diagonal matrices while $Z_{\mathfrak{M}}$ is the group of invertible scalar matrices, so there is no map from $[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M},H}]$ to $[\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}]$. In general, $Z_{\mathfrak{M},H}$ will be much larger than $Z_{\mathfrak{M}}$.

The solution to this problem is very simple: we always have a canonical subgroup $Z_{\mathfrak{M}}^{\kappa} \subset Z_{\mathfrak{M},H}$ that maps canonically onto $Z_{\mathfrak{M}}$. Essentially, $Z_{\mathfrak{M}}^{\kappa}$ is the "preimage" of $Z_{\mathfrak{M}}$ in $\mathfrak{M}_H$, which is necessarily contained in $Z_{\mathfrak{M},H}$. In our example above where $\mathfrak{M} = \mathrm{Mat}_2$ and $\mathfrak{M}_H = \mathbb{A}^2$, $Z_{\mathfrak{M}}^{\kappa}$ is equal to $Z_{\mathfrak{M}}$. In general, the group $Z_{\mathfrak{M}}^{\kappa}$ is again much larger than $Z_{\mathfrak{M}}$. We now have a map

$$[\mathfrak{C}_{\mathfrak{M},H}/Z_{\mathfrak{M}}^{\kappa}] \longrightarrow [\mathfrak{C}_{\mathfrak{M}}/Z_{\mathfrak{M}}], \tag{1.3.1}$$

compatible with the map between classifying stacks, $\mathbb{B}Z_{\mathfrak{M}}^{\kappa} \to \mathbb{B}Z_{\mathfrak{M}}$.

The map (1.3.1) is analogous to the map $[\mathfrak{c}_H/\mathbb{G}_{\mathrm{m}}] \to [\mathfrak{c}/\mathbb{G}_{\mathrm{m}}]$ in the Lie algebra case. Since for both $\mathfrak{g}$ and $\mathfrak{h}$ the twisting is given by $\mathbb{G}_{\mathrm{m}}$, we may pick a single $\mathbb{G}_{\mathrm{m}}$-torsor of large degree, as is done by most literature. In the group case, we cannot afford to fix any torsor because $Z_{\mathfrak{M}}^{\kappa}$ and $Z_{\mathfrak{M}}$ are different, and all we can do is to apply the mapping stack functor to the whole map (1.3.1). It turns out to be the correct formulation. In short, we obtain a finite unramified map

$$\nu_{\mathcal{A}}\colon \mathcal{A}_{H,X}^{\kappa} \longrightarrow \mathcal{A}_X,$$

which, unlike in the Lie algebra case, is usually *not* a closed embedding. Strictly speaking, here we need to replace $\mathcal{A}_X$ by an open subset (the reduced locus $\mathcal{A}_X^{\heartsuit}$), but we ignore such issues in this introduction for simplicity.

1.3.7. Using map (1.3.1), we will be able to match conjugacy classes of G and H. The transfer of functions is more complicated because it involves global L-embeddings. For simplicity, suppose we have an irreducible representation of the dual group $\check{\mathbf{G}}$ with highest weight λ; it decomposes into a direct sum of irreducible representations of $\check{\mathbf{H}}$. Note that for a $\check{\mathbf{H}}$-highest weight λ_H therein, it may have multiplicity greater than 1.

As it turns out, the choice of monoids $\mathfrak{M}$ and $\mathfrak{M}_H$, aside from conjugacy classes, will also match λ with the set of λ_H in the decomposition. We can then restore the multiplicity of λ_H by tensoring with an appropriate multiplicity space, viewed as a constant sheaf on $\mathsf{Gr}_H^{\le\lambda_H}$. We will refrain from talking about the technical details involving global L-embeddings, however. See § 9.3 for more details.

1.4 SUPPORT THEOREM

Continue following the same general strategy in [Ngô10] and its next and perhaps most crucial piece is the support theorem. We have already discussed several difficulties and features unique to the group case, and this part is no exception.

1.4.1. Ngô's support theorem in [Ngô10] has three main ingredients aside from the geometric transfer map already established. The first is the symmetry of Hitchin fibers, namely a Picard stack acting on the Hitchin fibration induced by the regular centralizer. Then from this symmetry one can stratify the Hitchin base in several ways. These two ingredients imply that Hitchin fibrations satisfy certain properties, making them so-called *δ-regular abelian fibrations*. Ngô then proves a support theorem for a general δ-regular abelian fibration with a smooth total stack. The support theorem at this level of generality is a very powerful tool, but on its own it is not quite enough for the fundamental lemma, notably because it establishes only an upper bound for the supports.

The reason that a general support theorem cannot pin down the exact set of supports is that the argument is ultimately based on dimension considerations. It is essentially a vastly optimized Goresky–MacPherson inequality (see [Ngô10, § 7.3]). As such, it cannot deal with other contributions that are discrete in nature. In the case of δ-regular abelian fibrations, this extra "discrete contribution" comes from the irreducible components of the fibers.

Ngô is able to determine the exact amount of such a discrete contribution for Hitchin fibrations using a concrete description of the irreducible components, and shows that the only such contribution comes from endoscopy.

1.4.2. In the group case, the first two ingredients remain largely unchanged. First, the regular centralizer can be defined similarly, albeit with more technical effort, so that there is a Picard action. Second, the stratifications on the mH-base can be done in a parallel fashion, showing that mH-fibrations are δ-regular abelian fibrations.

There is a new ingredient necessary for mH-fibrations: the local model of singularity. Unlike its connection to L-groups as we already discussed, its role here is rather simple to explain: suppose $f\colon X \to Y$ is a map of varieties over $\bar{k}$ of pure relative dimension d; then the cohomological degree of $\mathbf{R}f_!\overline{\mathbb{Q}}_\ell$ is bounded above by $2d$, and at degree $2d$ its stalks are spanned by irreducible components of the fibers of f. However, if we replace $\overline{\mathbb{Q}}_\ell$ by the intersection complex IC_X, then $\mathbf{R}f_!\mathrm{IC}_X$ is not necessarily cohomologically bounded above by $2d - \dim X$. Using a local model of singularity and a simple spectral sequence, we will show that such an upper bound does hold for mH-fibrations, and at the top degree the contribution can be described using irreducible components of the fibers. The idea of using a spectral sequence was originally due to A. Bouthier in an unpublished note.

1.4.3. With these inputs, we are able to prove a generalization of Ngô's support theorem for mH-fibrations, which reduces the problem to that of describing irreducible components of mH-fibers. Unlike in the Lie algebra case, the description of the irreducible components is much more complicated, and there are additional contributions, other than endoscopy. In order to understand it, we use a simple example to illustrate the phenomenon.

If we have two effective Cartier divisors $D' < D$ on X, we will have a natural inclusion of section spaces

$$\mathrm{H}^0(X, \mathcal{O}_X(D')) \subset \mathrm{H}^0(X, \mathcal{O}_X(D)).$$

Similarly, for (ordinary) Hitchin fibrations, we can embed a Hitchin base into another Hitchin base with larger degree:

$$\mathrm{H}^0(X, \mathfrak{c}_{D'}) \subset \mathrm{H}^0(X, \mathfrak{c}_D).$$

We call the image of this embedding an *inductive stratum* in the latter space. Of course, such strata are not very important in the Lie algebra case, but the story is different for mH-fibrations.

There is a similar partial order on boundary divisors in $\mathcal{B}_X$ induced by the dominance order on cocharacters. For a fixed boundary $b \in \mathcal{B}_X$, let $\mathcal{A}_b$ be the fiber of $\mathcal{A}_X$ over b; then we have a natural inclusion $\mathcal{A}_{b'} \subset \mathcal{A}_b$ for $b' < b$, defined in a similar way. The union of these linear subspaces from boundary subdivisors induces a stratification on $\mathcal{A}_b$, and we can let b vary in $\mathcal{B}_X$ to obtain a stratification on the whole $\mathcal{A}_X$. The strata are the inductive strata in mH-bases.

Consider the endoscopic locus $\mathcal{A}^\kappa \subset \mathcal{A}_X$ that is the union of the images of mH-bases $\mathcal{A}^\kappa_{H,X}$ for all endoscopic groups H associated with endoscopic parameter κ. Let S^κ_H be the set of images in $\mathcal{A}_X$ of inductive strata in $\mathcal{A}^\kappa_{H,X}$, so it is a set of closed subsets of $\mathcal{A}_X$. Now we can state a rough form of support theorem for mH-fibrations (see Theorems 10.9.3 and 10.9.7 for precise statements):

Theorem 1.4.4. *The complex $h_{X,*}\mathrm{IC}_{\mathcal{M}_X}$ is pure perverse and decomposes into a direct sum of κ-isotypic components $(h_{X,*}\mathrm{IC}_{\mathcal{M}_X})_\kappa$, where κ ranges over possible endoscopic parameters, and the set of supports of perverse sheaves ${}^{\mathrm{p}}\mathrm{H}^\bullet(h_{X,*}\mathrm{IC}_{\mathcal{M}_X})_\kappa$ is a subset of the union of the S^κ_H, where H ranges over all possible endoscopic groups associated with κ.*

1.4.5. As it turns out, the contribution of inductive strata can be described locally using *unramified* multiplicative affine Springer fibers, whose irreducible components have an explicit connection with MV-cycles. This leads to the final component of the proof of Theorem 1.0.1, which we discuss in the next section.

1.5 TRANSFER FACTOR AND KASHIWARA CRYSTALS

In [Ngô10], any problem involving the transfer factor is avoided because it has already been studied in [Kot99], and can be conveniently skipped over if one utilizes the Kostant section. In the group case, there does not seem to be any corresponding result. Hales gives a simplified definition of transfer factors for unramified groups in [Hal93], but it is still far more complicated than in the Lie algebra case in [Kot99]. In the case of unramified conjugacy classes, however, things can be more explicitly described by studying the Frobenius action on the set of irreducible components of multiplicative affine Springer fibers.

1.5.1. Recall that earlier we mentioned the proof of the fundamental lemma for unramified elements by [GKM04] that is conditional on the purity conjecture of affine Springer fibers. Even though it is still a conjecture at the time of writing, it is widely believed to be true, and it is reasonable to expect the same is true for their multiplicative analogues. Consequently, it is easy to believe that fundamental lemma can be more or less deduced from its asymptotic form by throwing away "lower order terms" and considering only the contribution from irreducible components. Of course, the latter is no longer a mere belief thanks to the support theorem, even though we do not have local purity results.

In [Chi22], Chi described the geometric irreducible components of unramified multiplicative affine Springer fibers by relating them to MV-cycles, and the latter give canonical bases of representations of the dual group. What is left is to describe the Frobenius action on those components and the κ-twisting. As MV-cycles form canonical bases in representations, a naive guess would be that the components on the G-side are related to those on the H-side via the restriction of representations from ${}^LG$ to ${}^LH$. This is, of course, too naive because it seems impossible to incorporate κ-twisting with the restriction functor.

1.5.2. The correct answer is to regard MV-cycles as basis elements not in representations, but in the associated Kashiwara crystals, or crystals for short. We will not review the theory of crystals here, but heuristically they capture the combinatorial essence of the highest-weight representations, and any highest-weight representation has a canonically associated (normal) crystal. One can sometimes think of a crystal basis as a particularly nicely chosen vector basis in the corresponding representation (although this point of view is not very accurate conceptually).

An important advantage of crystal bases is that they afford a canonical Weyl group action, while in contrast there is no Weyl group action on representations. Moreover, the restriction to standard Levi subgroups becomes very straightforward to describe. The trade-off, however, is that restriction functor in general becomes much more complicated: given a crystal and a (non-Levi) subgroup, there is no straightforward way to obtain a crystal of the subgroup by "restricting," and extra work needs to be done depending on the situation. This plays in our favor, because for endoscopic L-embeddings, this extra work is precisely the κ-twisted matching of irreducible components of multiplicative affine Springer fibers of G and of H.

1.5.3. The transfer factor defined in [LS87] is a product of five terms:

$$\Delta = \Delta_{\mathrm{I}}\Delta_{\mathrm{II}}\Delta_{\mathrm{III}_1}\Delta_{\mathrm{III}_2}\Delta_{\mathrm{IV}}.$$

The last term Δ_{IV} is just cohomological twist, and Δ_{III_1} is the coefficient for defining κ-orbital integrals, and is also relatively straightforward. The remaining terms, however, are rather complicated, and individually they depend on choices of certain auxiliary data. Fortunately, for unramified conjugacy classes, there are choices that are more or less canonical, and by using those choices, each individual term can be visualized in multiplicative affine Springer fibers.

As a result, we obtain an *asymptotic fundamental lemma* described using crystals. We will refrain from stating the precise result due to the complexity in notations (see Theorem 4.6.29). Even though this asymptotic version of the fundamental lemma looks a lot simpler than the original, it still seems quite difficult to prove. Nevertheless, there are special cases that are simple enough to prove by hand, and those cases are abundant enough for us to prove the fundamental lemma by induction. By reversing the argument, we can prove the asymptotic fundamental lemma in full as well.

1.5.4. It is interesting to note that in this book, the asymptotic fundamental lemma is deduced from the full fundamental lemma, even though the former looks much simpler. It would be interesting to find a purely local proof of the asymptotic fundamental lemma.

As a final note, we expect our framework of mH-fibrations can be generalized to a large class of spherical varieties hence used to tackle problems in relative trace formulae in general. For each case of such generalizations, we also expect that crystals will play a similar role in describing inner structures of L-packets, including but not limited to endoscopy.

1.6 STRUCTURE OF THIS BOOK

Here is a summary of the content in each chapter. Many chapters are arranged very much in the way of [Ngô10], not only because the logical structure therein is well organized, but also because it is more convenient to compare with the Lie algebra case.

In Chapter 2 we introduce some basic notations and setups in the absolute setting. We will review some standard facts about reductive monoids and their invariant theory under the adjoint action. The highlight of this chapter is the construction of the endoscopic monoid and the transfer map at the invariant-theoretic level. We will also give a formal statement of the fundamental lemma at the end.

Chapter 3 studies the multiplicative valuation strata. It is a purely technical tool for this book and is mostly self-contained and can be read by itself. Chapter 5 is also auxiliary. It collects many global constructions needed for our discussions, including the moduli of boundary divisors and a construction of global affine Schubert schemes using reductive monoids. The latter is somewhat interesting by itself as by using monoids we are able to define affine Schubert schemes without referring to affine Grassmannians.

Chapter 4 reviews constructions and properties of multiplicative affine Springer fibers for groups. Most of the chapter echoes what is done in the Lie algebra case, with two notable new additions: the connection with MV-cycles and the connection with Kashiwara crystals. We also discuss how to translate the fundamental lemma into the language of monoids.

Chapter 6 studies the constructions of mH-fibrations and their basic properties. It is more or less a global parallel of Chapter 4 and also echoes the Lie algebra case. The local model of singularity is also stated here, but the proof is postponed to Chapter 7, where we study the deformations of mHiggs bundles. In Chapter 7, we first review some general facts about deformations of mapping stacks, and then use them to study mH-bases. Using the certain duality between the mH-base and mH-fibers, we study the deformation of mHiggs bundles by modifying the argument for mH-bases.

In Chapter 8, we discuss various useful stratifications on the mH-base. The particularly new feature is the notion of inductive subsets, which will be important in our support theorem. In Chapter 9, we study cohomologies based on those stratifications. The global endoscopic transfer, both at the level of conjugacy classes and functions, in the form of global Satake sheaves, will be explained here. We will also state the geometric stabilization theorem. Another interesting object we introduce here is a new kind of Hecke-type stack, using which we can upgrade the traditional product formula over a point into a family. The latter will be particularly useful when we study the top ordinary cohomologies, as studying them beyond just the rank of stalks will be necessary.

Chapter 10 contains the proof of the support theorem. The proof itself is surprisingly close to that for the case of Lie algebra, with perhaps the only important new ingredient being the local model of singularity proved in Chapter 7. However, due to mH-fibers having a more complicated description of irreducible components, the implication of the support theorem differs from that in the Lie algebra case, and this is where inductive subsets and the connection with MV-cycles and crystals come into play.

In Chapter 11, we review the point-counting framework in [Ngô10, § 8], and extend it slightly to make it more convenient. After that, in Chapter 12, we will finally be able to prove the fundamental lemma after several reductions and inductions.

There are two appendices. In Appendix A we review the definition of the standard transfer factor defined in [LS87]. The language is slightly altered to be coherent with the main part of this book. In Appendix B we give some basic facts about Kashiwara crystals for the reader's convenience.

Chapter 2

Reductive Monoids and Invariant Theory

We start by reviewing some facts about the invariant theories of the adjoint action of a reductive group G on the simply-connected cover G^{sc} of its derived subgroup and on a very flat reductive monoid $\mathfrak{M}$ associated with G^{sc}. The new results in this chapter are mostly contained in § 2.5, where the canonical endoscopic monoid $\mathfrak{M}_H$ associated with any given monoid $\mathfrak{M}$ and endoscopic group H is defined. The endoscopic monoid plays a key role throughout this book. Proofs are mostly omitted for well-known facts, but references will be given when possible.

2.1 QUASI-SPLIT FORMS

In this section, we set up some basic notations used throughout this book. In particular, we introduce the notion of quasi-split forms.

2.1.1. Let k be a finite field with q elements and $\bar{k}$ be a fixed algebraic closure of k. Let $\mathbf{G}$ be a split connected reductive group defined over k of rank n_G and semisimple rank r. We assume $p = \mathrm{char}(k)$ is larger than twice the Coxeter number of $\mathbf{G}$. We fix once and for all a split maximal k-torus $\mathbf{T}$ of $\mathbf{G}$ and a Borel subgroup $\mathbf{B}$ containing $\mathbf{T}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $(\mathbb{X}, \Phi, \check{\mathbb{X}}, \check{\Phi})$ be the root datum associated with $(\mathbf{G}, \mathbf{T})$, and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ (resp. Φ_+) the set of simple (resp. positive) roots determined by $\mathbf{B}$, and let $\check{\Delta} = \{\check{\alpha}_1, \ldots, \check{\alpha}_r\}$ (resp. $\check{\Phi}_+$) be the simple (resp. positive) coroots. Let $\mathbb{X}_+$ (resp. $\check{\mathbb{X}}_+$) be the set of dominant characters (resp. cocharacters). Let $\mathbf{W} = W(\mathbf{G}, \mathbf{T})$ be the Weyl group, and let $w_0 \in \mathbf{W}$ be the longest element with respect to $\mathbf{B}$.

2.1.2. Let $\mathbf{G}^{\mathrm{der}}$ be the derived subgroup of $\mathbf{G}$, and let $\mathbf{G}^{\mathrm{sc}}$ (resp. $\mathbf{G}^{\mathrm{ad}}$) be the universal cover (resp. adjoint quotient) of $\mathbf{G}^{\mathrm{der}}$. We also use H^{sc} (resp. H^{ad}) to denote the preimage (resp. image) of any subset $H \subset \mathbf{G}$ in $\mathbf{G}^{\mathrm{sc}}$ (resp. $\mathbf{G}^{\mathrm{ad}}$). Let $\{\varpi_1, \ldots, \varpi_r\}$ be the set of fundamental weights of $\mathbf{G}^{\mathrm{sc}}$, and (ρ_i, V_{ϖ_i}) be the corresponding Weyl module with highest weight ϖ_i. Let $\{\check{\varpi}_1, \ldots, \check{\varpi}_r\}$ be the set of fundamental coweights. Let ρ (resp. $\check{\rho}$) be the half-sum of all positive roots (resp. coroots).

2.1.3. We fix a k-pinning $\wp = (\mathbf{T}, \mathbf{B}, \mathbf{x}_+ := \{\mathbf{U}_\alpha\}_{\alpha\in\Delta})$ of $\mathbf{G}$, where $\mathbf{U}_\alpha \colon \mathbb{G}_{\mathrm{a}} \to \mathbf{G}$ is a one-parameter unipotent group associated with simple root α. The associated groups $\mathbf{G}^{\mathrm{der}}$, $\mathbf{G}^{\mathrm{sc}}$, and $\mathbf{G}^{\mathrm{ad}}$ all come with pinnings induced by $(\mathbf{T}, \mathbf{B}, \mathbf{x}_+)$. Using this k-pinning, we may identify the group $\mathrm{Out}(\mathbf{G})$ of outer automorphisms of G with a subgroup of $\mathrm{Aut}_k(\mathbf{G})$. This is a discrete group, possibly infinite. Given any k-scheme X, we can consider étale $\mathrm{Out}(\mathbf{G})$-torsors over X.

2.1.4. Given such an $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G, we may obtain a quasi-split twisted form

$$G = \mathbf{G} \times^{\mathrm{Out}(\mathbf{G})} \vartheta_G$$

of $\mathbf{G}$ on X. This is a reductive group scheme over X together with a pinning $\wp = (T, B, x_+)$ relative to X. The torsor ϑ_G also induces an $\mathrm{Out}(\mathbf{G}^{\mathrm{ad}}) = \mathrm{Out}(\mathbf{G}^{\mathrm{sc}})$-torsor, still denoted by ϑ_G. It induces quasi-split forms G^{ad} and G^{sc} over X. The Weyl group $\mathbf{W}$ induces a Weyl group scheme W over X, which is an étale group scheme.

2.1.5. If we fix a geometric point $\infty \in X$, then any étale $\mathrm{Out}(\mathbf{G})$-torsor can be given by a continuous homomorphism

$$\vartheta_G^\bullet \colon \pi_1(X, \infty) \longrightarrow \mathrm{Out}(\mathbf{G}).$$

In this way the group G comes with a canonical geometric point ∞_G over ∞, and we use (G, ∞_G) for this pointed twisted form.

2.1.6. If G is *any* reductive group over an algebraically closed field with a fixed maximal torus T and Weyl group W, we define $\Xi_G(\lambda)$ for any $\lambda \in \mathbb{X}(T)$ to be the set of weights $\mu \in \mathbb{X}(T)$ in the convex hull of the orbit $W\lambda$ such that $\lambda - \mu$ is contained in the root lattice of T in G. If G is clear from the context, we will simply use $\Xi(\lambda)$.

2.2 INVARIANT THEORY OF THE GROUP

In this section, we review the invariant theory of the adjoint action of $\mathbf{G}$ on $\mathbf{G}^{\mathrm{sc}}$. Later, in § 2.4, we will extend most results in this section to reductive monoids.

2.2.1. The group $\mathbf{G}$ acts on $\mathbf{G}^{\mathrm{sc}}$ by adjoint action. Let $\chi \colon \mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathbf{G}$ be the GIT quotient map.

Theorem 2.2.2 ([Ste65]). *The inclusion $\mathbf{T}^{\mathrm{sc}} \hookrightarrow \mathbf{G}^{\mathrm{sc}}$ induces an isomorphism*

$$\mathbf{C} := \mathbf{T}^{\mathrm{sc}} /\!\!/ \mathbf{W} \xrightarrow{\sim} \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathbf{G}^{\mathrm{sc}} = \mathbf{G}^{\mathrm{sc}} /\!\!/ \mathbf{G}.$$

In addition, both schemes are isomorphic to affine space $\mathbb{A}^r$, whose coordinates are given by the traces χ_i of the fundamental representations (ρ_i, V_{ϖ_i}) of $\mathbf{G}^{\mathrm{sc}}$.

Definition 2.2.3. A $\mathbf{G}$-orbit $\mathbf{G}(\gamma)$ ($\gamma \in \mathbf{G}^{\mathrm{sc}}$) is called *regular* if its stabilizers have minimal dimension. It is called *semisimple* if it contains an element in $\mathbf{T}^{\mathrm{sc}}$. It is called *regular semisimple* if it is both regular and semisimple. The union of regular (resp. semisimple, resp. regular semisimple) orbits is denoted by $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ (resp. $\mathbf{G}^{\mathrm{sc}}_{\mathrm{ss}}$, resp. $\mathbf{G}^{\mathrm{sc}}_{\mathrm{rs}}$).

2.2.4. The regular locus $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is open and the restriction of χ to $\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is smooth and surjective. The semisimple locus is dense but not open in $\mathbf{G}^{\mathrm{sc}}$. Their intersection $\mathbf{G}^{\mathrm{sc}}_{\mathrm{rs}}$, however, remains open dense in $\mathbf{G}^{\mathrm{sc}}$, and its image in $\mathbf{C}$ is denoted by $\mathbf{C}^{\mathrm{rs}}$. Consider the *discriminant function* on $\mathbf{T}$:

$$\mathrm{Disc} := \prod_{\alpha\in\Phi}(1 - e^{\alpha}).$$

This function is $\mathbf{W}$-invariant, hence descends to a function on $\mathbf{C}$, still denoted by Disc. It defines an effective principal divisor $\mathbf{D}$ on $\mathbf{C}$, called the *discriminant divisor*. In fact, $\mathbf{D}$ is a reduced divisor, so it makes sense to talk about its singular locus $\mathbf{D}^{\mathrm{sing}}$.

Proposition 2.2.5 ([Ste65])**.** *The regular semisimple locus* $\mathbf{C}^{\mathrm{rs}}$ *is equal to the complement of the divisor* $\mathbf{D}$ *and thus is an open subset of* $\mathbf{C}$*. Moreover, it is exactly the locus over which the fiber of* χ *consists of a single* $\mathbf{G}$*-orbit.*

2.2.6. The quotient map χ admits many sections, as in the Lie algebra case. The difference is that the section we want to use lacks an explicit formula, unlike, for example, the Kostant section for Lie algebras. Instead, what Steinberg explicitly constructed is a quasi-section. A morphism f in a category is called a *quasi-section* to morphism g if $g \circ f$ is an *automorphism* (not necessarily identity).

Definition 2.2.7. Fix our choice of simple roots Δ from earlier. A *Coxeter datum* is a pair $(\xi, \dot{S})$, where

(i) $\xi\colon \{1, \ldots, r\} \to \Delta$ is a bijection (i.e., a total ordering on the set of simple roots),
(ii) $\dot{S}$ is a set of representatives $\dot{s}_\alpha$ in $\mathrm{N}_{\mathbf{G}}(\mathbf{T})$ of simple reflections determined by Δ.

A *Coxeter element* (after fixing a set of simple reflections) of $\mathbf{W}$ is one that can be written as $w = w_\xi = s_{\xi(1)} \cdots s_{\xi(r)}$ for some ξ. Denote by $\mathrm{Cox}(\mathbf{W}, \Delta)$ the set of all Coxeter elements of $\mathbf{W}$.

Fix a Coxeter datum $(\xi, \dot{S})$. Let $\beta_j = \xi(j)$ be the simple root corresponding to $s_{\xi(j)}$. Recall we have one-parameter groups $\mathbf{U}_\alpha$ in the pinning $\wp$, such that $\mathrm{Ad}_t(\mathbf{U}_\alpha(x)) = \mathbf{U}_\alpha(\alpha(t)x)$ for all $t \in \mathbf{T}$ and $x \in \mathbb{G}_{\mathrm{a}}$. Let

$$\begin{aligned} \epsilon^{(\xi,\dot{S})}\colon \mathbf{C} \cong \mathbb{A}^r &\longrightarrow \mathbf{G}^{\mathrm{sc}}, \\ (x_1, \ldots, x_r) &\longmapsto \prod_{j=1}^{r} \mathbf{U}_{\beta_j}(x_j)\dot{s}_{\xi(j)}, \end{aligned}$$

where the product on the right-hand side is considered taken in the specified order. This is the Steinberg quasi-section associated with Coxeter datum $(\xi, \dot{S})$. We also call the image $\operatorname{Im}\epsilon^{(\xi,\dot{S})}$ a *Steinberg cross-section.* We summarize the results in the following theorem.

Theorem 2.2.8 ([Ste65]). *For each pair $(\xi, \dot{S})$, the map $\epsilon^{(\xi,\dot{S})}$ is a quasi-section of χ. Moreover, $\operatorname{Im}\epsilon^{(\xi,\dot{S})}$ is contained in the regular locus, and has transversal intersection with each regular orbit.*

Once $\wp$ is fixed, the construction of Steinberg quasi-section relies on two choices: a total ordering of the simple roots, and representatives of simple reflections. The influence of the choices is summarized below.

Proposition 2.2.9 ([Ste65, Lemma 7.5 and Proposition 7.8]). *For any Coxeter data $(\xi, \dot{S})$ and $(\xi', \dot{S}')$:*

(i) If $\xi = \xi'$, then there exists $t, t' \in \mathbf{T}$ such that $\operatorname{Im}\epsilon^{(\xi,\dot{s}')} = t' \operatorname{Im}\epsilon^{(\xi,\dot{S})} = t \operatorname{Im}\epsilon^{(\xi,\dot{S})} t^{-1}$.

(ii) If $\dot{S} = \dot{S}'$, then for any $x, x' \in \mathbb{A}^r$ such that $x_{\xi(j)} = x'_{\xi'(j)}$ for $1 \le j \le r$, $\epsilon^{(\xi,\dot{S})}(x)$ and $\epsilon^{(\xi',\dot{S})}(x')$ are $\mathbf{G}$-conjugate. In fact, such conjugation can be made functorially for any k-algebra R. In other words, the transporter from $\epsilon^{(\xi,\dot{S})}(\mathbf{C})$ to $\epsilon^{(\xi',\dot{S})}(\mathbf{C})$ is a trivial torsor under the centralizer group scheme over $\epsilon^{(\xi,\dot{S})}(\mathbf{C})$.

2.2.10. The universal centralizer group scheme $\mathbf{I} \to \mathbf{G}^{\mathrm{sc}}$ restricts to a smooth group scheme over the regular locus $\mathbf{I}^{\mathrm{reg}} \to \mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$. Since generically the centralizer is a torus, $\mathbf{I}^{\mathrm{reg}} \to \mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}$ is a commutative group scheme. Therefore, one can utilize the descent argument in [Ngô10, Lemme 2.1.1] to obtain the following result.

Proposition 2.2.11. *There is a unique smooth commutative group scheme $\mathbf{J} \to \mathbf{C}$ with a $\mathbf{G}$-equivariant isomorphism*

$$\chi^* \mathbf{J}|_{\mathbf{G}^{\mathrm{sc}}_{\mathrm{reg}}} \xrightarrow{\sim} \mathbf{I}^{\mathrm{reg}},$$

which can be extended to a homomorphism $\chi^* \mathbf{J} \to \mathbf{I}$.

Definition 2.2.12. The group scheme $\mathbf{J} \to \mathbf{C}$ is called the *regular centralizer.*

There is another description of $\mathbf{J}$ using the cameral cover $\pi\colon \mathbf{T}^{\mathrm{sc}} \to \mathbf{C}$, similarly to the Lie algebra case in [DG02] and [Ngô10]. Consider the trivial family of torus $\mathrm{pr}_2\colon \mathbf{T} \times \mathbf{T}^{\mathrm{sc}} \to \mathbf{T}^{\mathrm{sc}}$ (pr_2 means the second projection), and we have Weil restriction

$$\Pi_{\mathbf{G}} := \pi_*(\mathbf{T} \times \mathbf{T}^{\mathrm{sc}}),$$

on which $\mathbf{W}$ acts diagonally. Since π is finite flat, $\Pi_{\mathbf{G}}$ is representable. Moreover, since $\mathbf{T} \times \mathbf{T}^{\mathrm{sc}}$ is smooth, so is Π_G. Over $\mathbf{C}^{\mathrm{rs}}$, π is Galois étale with Galois group $\mathbf{W}$;

hence $\Pi_{\mathbf{G}}^{\mathrm{rs}}$ is a torus. Since $\mathrm{char}(k)$ does not divide the order of $\mathbf{W}$, we have a smooth group scheme $\mathbf{J}^1$ over $\mathbf{C}$:

$$\mathbf{J}^1 := \Pi_{\mathbf{G}}^{\mathbf{W}}.$$

Let $\mathbf{J}^0 \subset \mathbf{J}^1$ be the open subgroup scheme consisting of fiberwise neutral components.

Similarly to [Ngô10, Définition 2.4.5], we consider this subfunctor $\mathbf{J}'$ of $\mathbf{J}^1$: for a $\mathbf{C}$-scheme S, $\mathbf{J}'(S)$ consists of points

$$f : S \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}} \to \mathbf{T}$$

such that for every geometric point $x \in S \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}}$, if $s_\alpha(x) = x$ for a root α, then $\alpha(f(x)) \neq -1$. Notice that on $\mathbf{T}^{\mathrm{sc}}$, the condition $s_\alpha(x) = x$ is the same as $x \in \ker \alpha$.

Lemma 2.2.13. *The subfunctor $\mathbf{J}'$ is representable by an open subgroup scheme of $\mathbf{J}^1$ containing $\mathbf{J}^0$.*

Proof. The proof is entirely parallel to [Ngô10, Lemme 2.4.6]. Indeed, it suffices to prove this claim after a finite flat base change to $\mathbf{T}^{\mathrm{sc}}$.

On $\mathbf{T}^{\mathrm{sc}}$, the discriminant divisor is the union (with multiplicities) of subgroups $\ker \alpha \subset \mathbf{T}^{\mathrm{sc}}$ for all roots $\alpha \in \Phi$. By adjunction, we have a map $\mathbf{J}^1 \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}} \to \mathbf{T} \times \mathbf{T}^{\mathrm{sc}}$, whose restriction to $\ker \alpha$ factors through fixed-point subgroup $\mathbf{T}^{s_\alpha} \times \ker \alpha$. Therefore, we have a map

$$\mathbf{J}^1 \times_{\mathbf{C}} \ker \alpha \xrightarrow{\alpha} \{\pm 1\}.$$

The inverse image of -1 is an open and closed subset of $\mathbf{J}^1 \times_{\mathbf{C}} \ker \alpha$, hence a Cartier divisor on $\mathbf{J}^1 \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}}$. The subfunctor $\mathbf{J}' \times_{\mathbf{C}} \mathbf{T}^{\mathrm{sc}}$ is the complement of these Cartier divisors, hence an open subgroup scheme. In addition, $\mathbf{J}'$ contains $\mathbf{J}^0$. ∎

Proposition 2.2.14. *There exists a canonical open embedding of group schemes $\mathbf{J} \to \mathbf{J}^1$ that identifies $\mathbf{J}$ with subgroup scheme $\mathbf{J}'$.*

Proof. The claim about open embedding is proved in [Chi22, Proposition 2.3.2]. In fact, the argument in [Chi22] shows that if $\mathbf{G} = \mathbf{G}^{\mathrm{sc}}$, then $\mathbf{J} \to \mathbf{J}^1$ is an isomorphism. Moreover, since on $\mathbf{T}^{\mathrm{sc}}$, $\ker \alpha$ and $\mathbf{T}^{\mathrm{sc}, s_\alpha}$ coincide for any root α, we also see that $\mathbf{J}' = \mathbf{J}^1$.

In general, we have a canonical map $\mathbf{J}^{\mathrm{sc}} \to \mathbf{J}$ and $\mathbf{J}$ is generated by the image of $\mathbf{J}^{\mathrm{sc}}$ and $\mathbf{Z}_{\mathbf{G}}$. On the other hand, the map $\mathbf{T}^{\mathrm{sc}} \to \mathbf{T}$ induces canonical map $\mathbf{J}^{\mathrm{sc},1} \to \mathbf{J}^1$ compatible with the map $\mathbf{J}^{\mathrm{sc}} \to \mathbf{J}$. By definition of $\mathbf{J}'$ we also have a third map $\mathbf{J}^{\mathrm{sc},\prime} \to \mathbf{J}'$. This means that the image of $\mathbf{J}^{\mathrm{sc},1}$ is contained in $\mathbf{J}'$. Since on $\mathbf{Z}_{\mathbf{G}}$ all roots are trivial, we know that the image of $\mathbf{J}$ is contained in $\mathbf{J}'$.

It remains to show that $\mathbf{J} \to \mathbf{J}'$ is an isomorphism. Then we can repeat the "codimension 2" argument of [Chi22, Proposition 2.3.2] (see also [Ngô10, Proposition 2.4.7]) to reduce the problem to the case where $\mathbf{G} = \mathrm{SL}_2$, GL_2, or PGL_2. The direct calculation is omitted. ∎

Remark 2.2.15. The proof we present here is subtly different from [Ngô10, Proposition 2.4.7], where there is a surjection $\pi_0(\mathbf{Z}_{\mathbf{G}}) \to \pi_0(\mathbf{J}_{\mathfrak{g}}) = \mathbf{J}_{\mathfrak{g}}/\mathbf{J}_{\mathfrak{g}}^0$ (here $\mathbf{J}_{\mathfrak{g}}$ stands for the regular centralizer for the Lie algebra of $\mathbf{G}$). It is not so in group case. For example, suppose $\mathbf{G}$ is simple of type G_2. Then $\mathbf{G} = \mathbf{G}^{\mathrm{ad}} = \mathbf{G}^{\mathrm{sc}}$. Let $x \in \mathbf{T}$ be such that $C_{\mathbf{G}}(x) \cong SL_3$. Let $u \in U_3 := \mathbf{U} \cap SL_3$ be a regular unipotent element in SL_3. Assuming $\mathrm{char}(k)$ is large enough, $C_{\mathbf{G}}(xu)$ contains $C_{SL_3}(u) = Z_{SL_3} \times C_{U_3}(u)$ with finite index. So $C_{\mathbf{G}}(xu)$ is disconnected but $\mathbf{Z}_{\mathbf{G}}$ is trivial.

2.2.16. Let G be a quasi-split form of $\mathbf{G}$ over k-scheme X, induced by an $\mathrm{Out}(\mathbf{G})$-torsor ϑ_G. We can twist almost every construction in the invariant theory of $\mathbf{G}$ by ϑ_G and obtain a twisted form over X. At first we still have adjoint action of G on G^{sc}, with invariant quotient $\chi\colon G^{\mathrm{sc}} \to \mathfrak{C} = G^{\mathrm{sc}} /\!\!/ G$. We also have the natural isomorphism

$$T /\!\!/ W \xrightarrow{\sim} G^{\mathrm{sc}} /\!\!/ G.$$

The discriminant is invariant under $\mathrm{Out}(\mathbf{G})$, so we have a divisor $\mathfrak{D} \subset \mathfrak{C}$ relative to X. The open loci of regular and regular semisimple orbits are stable under $\mathrm{Out}(\mathbf{G})$, and so we still have the twisted form of regular centralizer $\mathfrak{J} \to \mathfrak{C}$ over X, as well as its Galois construction $\mathfrak{J}^1$.

2.2.17. An important difference here compared to the Lie algebra case is that Steinberg quasi-section does not necessarily exist, since it may not be stable under $\mathrm{Out}(\mathbf{G})$. Nevertheless, one may obtain a weaker result as follows. The Steinberg quasi-sections depends on a choice of representatives of simple reflections in $\mathbf{G}^{\mathrm{sc}}$. Using the pinning $\wp$, one can make such a choice that is stable under $\mathrm{Out}(\mathbf{G})$. Indeed, the root vectors in $\mathbf{x}_+$ can each be extended to a unique $\mathfrak{sl}_2$-triple; hence we get an opposite pinning with root vectors denoted by $\mathbf{x}_-$. For each simple root $\alpha_i \in \Delta$ (under $\mathbf{B}$), we let

$$\dot{s}_i = \mathbf{U}_{\alpha_i}(1)\mathbf{U}_{-\alpha_i}(-1)\mathbf{U}_{\alpha_i}(1),$$

where the $\mathbf{U}_{\alpha_i}$ (resp. $\mathbf{U}_{-\alpha_i}$) are the one-parameter unipotent subgroups determined by $\mathbf{x}_+$ (resp. $\mathbf{x}_-$).

The Steinberg quasi-section also depends on a choice of ordering on Δ. When $\mathbf{G}^{\mathrm{sc}}$ does not contain any simple factor of type A_{2m}, any two simple roots conjugate under $\mathrm{Out}(\mathbf{G})$ are not linked in the Dynkin diagram. Therefore, by grouping together the $\mathrm{Out}(\mathbf{G})$ orbits when making the ordering, we can make such a Steinberg quasi-section equivariant under $\mathrm{Out}(\mathbf{G})$. Thus, a section to χ always exists as long as $\mathbf{G}^{\mathrm{sc}}$ does not have any simple factor of type A_{2m}.

In fact, we have a slightly stronger result: let $\mathbf{G}'$ be the direct factor of $\mathbf{G}^{\mathrm{sc}}$ consisting of all its simple factors of type A_{2m} (for various m), which is preserved by $\mathrm{Out}(\mathbf{G})$. Then a Steinberg quasi-section exists for G^{sc} as long as the twist G' of $\mathbf{G}'$ is split.

In the remaining cases (i.e., G' is a nontrivial outer twist), we cannot hope to construct a section whose image lies in the regular locus. Rather, we have a weaker result by Steinberg, which is still very helpful later.

Theorem 2.2.18 ([Ste65, Theorem 9.8])**.** *Let G be a quasi-split semisimple and simply-connected group over a perfect field K; then $G(K) \to \mathfrak{C}(K)$ is surjective.*

The statement in [Ste65] requires K to be perfect, but the proof in fact works for an arbitrary field provided that $\mathrm{char}(K)$ is not too small for G (e.g., is larger than twice the Coxeter number of G). If K is perfect, then the result refines to $G^{\mathrm{ss}}(K) \to \mathfrak{C}(K)$ being surjective. We will postpone the details until Theorem 2.4.27 since we need to extend the above result to reductive monoids.

2.3 REVIEW OF VERY FLAT REDUCTIVE MONOIDS

The Lie algebra of $\mathbf{G}$ carries a natural $\mathbb{G}_{\mathrm{m}}$-action from its vector space structure, which is useful in global constructions. The group $\mathbf{G}^{\mathrm{sc}}$, however, has no such symmetry built in. Therefore, we must embed $\mathbf{G}^{\mathrm{sc}}$ into a $\mathbf{G}$-space where a similar "scaling" is possible. The quintessential example of this is the embedding of SL_n into Mat_n. In general, we use the theory of very flat reductive monoids.

2.3.1. An algebraic semigroup over k is just a k-scheme of finite type M together with a multiplication morphism $m\colon M \times M \to M$, such that the usual commutative diagram of associativity holds. If there exists a multiplicative identity $e\colon \operatorname{Spec} k \to M$, then M is an algebraic monoid over k. We will only consider monoids that are affine, integral and normal as schemes. If the subgroup $M^{\times}$ of invertible elements of M is a reductive group, then we call M a *reductive monoid*. In this case, we denote the derived subgroup of $M^{\times}$ by M^{der}, and call it the *derived subgroup* of M.

Example 2.3.2. A (normal) toric variety M is a reductive monoid, whose unit group $M^{\times}$ is the torus acting on it, and whose derived subgroup is the trivial group. The variety Mat_n with matrix multiplication is also a reductive monoid, whose unit group is GL_n and derived subgroup is SL_n.

The category of normal reductive monoids is classified by Renner (see, e.g., [Ren05]) with the help of their unit groups. It is well known that over an algebraically closed field K, the category of normal affine toric varieties A of a fixed torus T is equivalent to that of strictly convex and saturated cones $\mathscr{C} \subset \check{\mathbb{X}}(T)$. If $T \subset G$ is a maximal torus in a reductive group over K with Weyl group W, and $\mathscr{C}$ is stable under W-action, then the W-action extends over A. Renner's classification theorem states that reductive monoids M with unit group G are classified by such cones, $\mathscr{C}$. More precisely, we have the following result.

Theorem 2.3.3 ([Ren05, Theorems 5.2 and 5.4])**.** *Let G be a reductive group over an algebraically closed field with maximal torus T and Weyl group $W = W(G,T)$.*

(i) Let $T \subset A$ be a normal affine toric variety such that the W-action on T extends over A. Then there exists a normal monoid M with unit group G such that $\overline{T}$ (the Zariski closure of T in M) is isomorphic to A.

(ii) *Let M be any normal reductive monoid with unit group G, then the submonoid $\overline{T}$ is normal.*

(iii) *If M_1 and M_2 are such that we have a commutative diagram of monoids*

$$\begin{tikzcd} \overline{T}_1 \arrow[d] & T_1 \arrow[l] \arrow[r] \arrow[d] & G_1 \arrow[d] \\ \overline{T}_2 & T_2 \arrow[l] \arrow[r] & G_2, \end{tikzcd}$$

then this diagram extends to a unique homomorphism $M_1 \to M_2$. Moreover, if the vertical arrows are isomorphisms, then $M_1 \cong M_2$.

Remark 2.3.4. The result is proved over an algebraically closed field, but it is not hard to see that if G is split over a perfect field k, T is a split maximal torus, and all the maps in the commutative diagram are defined over k, then $M_1 \to M_2$ is also defined over k by looking at $\mathrm{Gal}(\overline{k}/k)$-action. The existence of M given G and A is proved using k-rational representations which are well defined since G is split. See [Ren05] for details.

2.3.5. We first give an explicit description of a very important monoid $\mathbf{M}$ with $\mathbf{M}^{\mathrm{der}} \simeq \mathbf{G}^{\mathrm{sc}}$. Consider the group

$$\mathbf{G}_+ := (\mathbf{T}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}})/\mathbf{Z}^{\mathrm{sc}},$$

where the center $\mathbf{Z}^{\mathrm{sc}}$ of $\mathbf{G}^{\mathrm{sc}}$ acts on $\mathbf{T}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$ anti-diagonally. There is a maximal torus $\mathbf{T}_+ = (\mathbf{T}^{\mathrm{sc}} \times \mathbf{T}^{\mathrm{sc}})/\mathbf{Z}^{\mathrm{sc}}$ in $\mathbf{G}_+$, and we denote $\mathbf{Z}_+ = \mathbf{Z}_{\mathbf{G}_+} \cong \mathbf{T}^{\mathrm{sc}}$. The character lattice of $\mathbf{T}_+$ is identified as

$$\mathbb{X}(\mathbf{T}_+) = \big\{(\lambda, \mu) \in \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) \times \mathbb{X}(\mathbf{T}^{\mathrm{sc}}) \,\big|\, \lambda - \mu \in \mathbb{X}(\mathbf{T}^{\mathrm{ad}})\big\},$$

and its cocharacter lattice is identified as

$$\check{\mathbb{X}}(\mathbf{T}_+) = \big\{(\check{\lambda}, \check{\mu}) \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}}) \times \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}}) \,\big|\, \check{\lambda} + \check{\mu} \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{sc}})\big\}.$$

So we have characters $(\alpha, 0)$ $(\alpha \in \Delta)$ of $\mathbf{T}_+$ which are also one-dimensional representations of $\mathbf{G}_+$. On the other hand, any k-rational representation ρ of highest weight λ of $\mathbf{G}^{\mathrm{sc}}$ extends to $\mathbf{G}_+$ by the formula

$$\rho_+(z, g) = \lambda(z)\rho(g),$$

which is easily seen as well-defined. Consider the representation of $\mathbf{G}_+$

$$\rho_\star := \bigoplus_{i=1}^{r} (\alpha_i, 0) \oplus \bigoplus_{i=1}^{r} \rho_{i+}.$$

We take the normalization of the closure of $\mathbf{G}_+$ in $\rho_\star$, and denote it by $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$. This is the *universal monoid* associated with $\mathbf{G}^{\mathrm{sc}}$, constructed by Vinberg ([Vin95]) in characteristic 0 and Rittatore ([Rit01]) in all characteristics. We note that Rittatore's construction is based on the theory of spherical embeddings. To see that the description here coincides with Rittatore's, we use Theorem 2.3.3, part (iii), and it is straightforward to see that the maximal toric varieties in both cases coincide (cf. § 2.3.17).

2.3.6. Given any reductive monoid $\mathbf{M}$ with derived subgroup $\mathbf{G}^{\mathrm{sc}}$, we have an action of $\mathbf{G}^{\mathrm{sc}}$ by multiplication on the left, and another one by inverse-multiplication on the right. These two actions commute, hence combine to a $\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$-action on $\mathbf{M}$. The GIT quotient

$$\alpha_{\mathbf{M}} \colon \mathbf{M} \longrightarrow \mathbf{A}_{\mathbf{M}} := \mathbf{M} /\!\!/ (\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}})$$

is called the *abelianization* of $\mathbf{M}$, in the sense that it is the largest quotient monoid of $\mathbf{M}$ that is commutative.

Theorem 2.3.7 ([Vin95, Theorem 3], [Rit01, Theorem 2])**.** *Let $\mathbf{Z}_{\mathbf{M}}$ be the center of $\mathbf{M}^\times$, $\mathbf{Z}_{\mathbf{M},0}$ its neutral component, and $\mathbf{Z}_0 = \mathbf{Z}_{\mathbf{M},0} \cap \mathbf{G}^{\mathrm{sc}}$; then we have that*

(i) $\mathbf{A}_{\mathbf{M}}$ is a normal commutative monoid with unit group $\alpha_{\mathbf{M}}(\mathbf{Z}_{\mathbf{M}}) \simeq \mathbf{Z}_{\mathbf{M},0}/\mathbf{Z}_0$,
(ii) $\alpha_{\mathbf{M}}^{-1}(1) = \mathbf{G}^{\mathrm{sc}}$ and $\alpha_{\mathbf{M}}^{-1}(\alpha_{\mathbf{M}}(\mathbf{Z}_{\mathbf{M},0})) = \mathbf{G}_+$,
(iii) $\alpha_{\mathbf{M}}(\overline{\mathbf{Z}_{\mathbf{M},0}}) = \mathbf{A}_{\mathbf{M}}$ and $\mathbf{A}_{\mathbf{M}} \simeq \overline{\mathbf{Z}_{\mathbf{M},0}} /\!\!/ \mathbf{Z}_0$, where $\overline{\mathbf{Z}_{\mathbf{M},0}}$ denotes the Zariski closure of $\mathbf{Z}_{\mathbf{M},0}$ in $\mathbf{M}$.

2.3.8. Given a homomorphism of two reductive monoids $\phi\colon \mathbf{M}' \to \mathbf{M}$, by the universal property of the GIT quotient it induces a homomorphism $\phi_{\mathbf{A}} \colon \mathbf{A}_{\mathbf{M}'} \to \mathbf{A}_{\mathbf{M}}$ that fits into the commutative square

$$\begin{tikzcd} \mathbf{M}' \arrow[r, "\phi"] \arrow[d, "\alpha_{\mathbf{M}'}"'] & \mathbf{M} \arrow[d, "\alpha_{\mathbf{M}}"] \\ \mathbf{A}_{\mathbf{M}'} \arrow[r, "\phi_{\mathbf{A}}"] & \mathbf{A}_{\mathbf{M}}. \end{tikzcd} \tag{2.3.1}$$

Definition 2.3.9. The homomorphism $\phi\colon \mathbf{M}' \to \mathbf{M}$ is called *excellent* if (2.3.1) is Cartesian.

Definition 2.3.10. A reductive monoid $\mathbf{M}$ is called *very flat* if $\alpha_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{A}_{\mathbf{M}}$ is flat with integral fibers. Let $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ be the category in which:

(i) objects are very flat reductive monoids $\mathbf{M}$ with $\mathbf{M}^{\mathrm{der}} \cong \mathbf{G}^{\mathrm{sc}}$;
(ii) morphisms from $\mathbf{M}'$ to $\mathbf{M}$ are excellent homomorphisms $\phi\colon \mathbf{M}' \to \mathbf{M}$.

Let $\mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$ be the full subcategory of $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ whose objects are very flat monoids with element 0 (one such that $0x = x0 = 0$ for all $x \in \mathbf{M}$).

Theorem 2.3.11 ([Vin95, Theorem 5], [Rit01, Theorem 9]). *If* $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$*, then*

$$\mathrm{Hom}_{\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})}(\mathbf{M}, \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})) \neq \varnothing$$

and is a singleton if $\mathbf{M} \in \mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$*. In other words,* $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ *is a versal (resp. universal) object in* $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ *(resp.* $\mathcal{FM}_0(\mathbf{G}^{\mathrm{sc}})$*).*

Remark 2.3.12. Theorem 2.3.11 implies that a homomorphism $\mathbf{M} \to \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ will induce a homomorphism $\phi_{\mathbf{Z_M}} : \mathbf{Z_M} \to \mathbf{T}^{\mathrm{sc}}$ whose restriction on $\mathbf{Z}^{\mathrm{sc}}$ is the identity. In the case that $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, $\phi_{\mathbf{Z_M}} = \mathrm{id}_{\mathbf{T}^{\mathrm{sc}}}$.

2.3.13. The abelianization of the universal monoid $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ is an affine space of dimension r, whose coordinate functions can be exactly given by $e^{(\alpha,0)}$, where the $\alpha \in \Delta$ are simple roots. In the language of toric varieties, $\mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is the $\mathbf{T}^{\mathrm{ad}}$-toric variety associated with the cone generated by fundamental coweights $\check{\varpi}_i \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$.

If $\mathbf{A}$ is any affine normal toric variety, and $\phi\colon \mathbf{A} \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ is a homomorphism of algebraic monoids, then the pullback of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ through ϕ gives an object $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ with abelianization $\mathbf{A_M} = \mathbf{A}$, and by Theorem 2.3.11, every $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ arises in this way.

An important class of monoids consists of those corresponding to the map $\mathbb{A}^1 \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$ given by a *dominant* cocharacter $\lambda \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$. We will denote such a monoid by $\mathbf{M}(\lambda)$. More generally, we may also consider the monoid formed using a tuple of dominant cocharacters $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$ (allowing repetitions), or, equivalently, a multiplicative map $\mathbb{A}^m \to \mathbf{A}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$. We shall denote this monoid by $\mathbf{M}(\underline{\lambda})$.

Moreover, if $\underline{\lambda}$ is a tuple of dominant cocharacters in $\check{\mathbb{X}}(\mathbf{T})$, it induces a tuple $\underline{\lambda}_{\mathrm{ad}}$ of dominant cocharacters in $\check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})$. The induced map $\mathbb{A}^m \to \mathbf{A}_{\mathrm{Env}(G^{\mathrm{sc}})}$ also produces a monoid, still denoted by $\mathbf{M}(\underline{\lambda})$. When $\underline{\lambda}$ is a singleton, $\mathbf{M}(\underline{\lambda})$ is often called an *L-monoid* in the literature.

Example 2.3.14. When $\mathbf{G}^{\mathrm{sc}} = \mathrm{SL}_n$ and $\mathbf{M} = \mathrm{Mat}_n$, the abelianization map is the determinant, and the excellent map $\mathbf{M} \to \mathrm{Env}(\mathrm{SL}_n)$ is the pullback of

$$\mathbf{A_M} \cong \mathbb{A}^1 \longrightarrow \mathbf{A}_{\mathrm{Env}(\mathrm{SL}_n)} \cong \mathbb{A}^{n-1}$$

corresponding to cocharacter $\check{\varpi}_{n-1}$, and the map $\phi_{\mathbf{Z_M}}$ is the cocharacter $\check{\alpha}_1 + 2\check{\alpha}_2 + \cdots + (n-1)\check{\alpha}_{n-1} = n\check{\varpi}_{n-1}$. In other words, $\mathrm{Mat}_n = \mathbf{M}(\check{\varpi}_{n-1})$.

2.3.15. It is sometimes convenient to define a *numerical boundary divisor* $\mathbf{E_M}$ on $\mathbf{A_M}$ as the complement of $\mathbf{A}_{\mathbf{M}}^{\times}$, with reduced scheme structure. It is a Weil divisor, and when $\mathbf{A_M}$ is factorial, it is an effective Cartier divisor. In this book we will mostly be interested in those $\mathbf{M}$ such that $\mathbf{A_M}$ is isomorphic to an affine space $\mathbb{A}^m$, in which case $\mathbf{E_M}$ is cut out by the product of the m coordinate functions, and hence is principal. Note that $\mathbf{E_M}$ is in general *not* the pullback of $\mathbf{E}_{\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})}$, but it always contains the latter after passing to the underlying topological spaces.

2.3.16. The abelianization map admits a section as follows: for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, let $\mathbf{T}^{\mathrm{ad}} \to \mathbf{T}_+$ be the map $t \mapsto (t, t^{-1})$. It is well-defined and extends to a map $\delta_{\mathbf{M}}\colon \mathbf{A}_{\mathbf{M}} \to \mathbf{M}$, which is in fact a section of $\alpha_{\mathbf{M}}$. For a general monoid in $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$, the formula is $z \mapsto (z, \phi_{\mathbf{Z}_{\mathbf{M}}}(z)^{-1})$.

2.3.17. Let $\mathbf{T}_{\mathbf{M}}$ be the maximal torus of $\mathbf{M}^{\times}$ containing $\mathbf{T}^{\mathrm{sc}}$, and $\overline{\mathbf{T}}_{\mathbf{M}}$ its closure in $\mathbf{M}$. It is a normal affine toric variety under $\mathbf{T}_{\mathbf{M}}$, and when $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, its character cone $\mathscr{C}^*_{\overline{\mathbf{T}}_{\mathbf{M}}}$ and cocharacter cone $\mathscr{C}_{\overline{\mathbf{T}}_{\mathbf{M}}}$ have the following description:

$$\begin{aligned}
\mathscr{C}^*_{\overline{\mathbf{T}}_{\mathbf{M}}} &= \left\{(\lambda,\mu) \in \mathbb{X}(\mathbf{T}_+) \,\middle|\, \lambda \geq w(\mu), w \in \mathbf{W}\right\}\\
&= \mathbb{N}\left(\left\{(\alpha,0) \,\middle|\, \alpha \in \Delta\right\} \cup \left\{(\varpi_i, w(\varpi_i)) \,\middle|\, 1 \leq i \leq r \text{ and } w \in \mathbf{W}\right\}\right),\\
\mathscr{C}_{\overline{\mathbf{T}}_{\mathbf{M}}} &= \left\{(\check{\lambda},\check{\mu}) \in \check{\mathbb{X}}(\mathbf{T}_+) \,\middle|\, \check{\lambda} \in \check{\mathbb{X}}(\mathbf{T}^{\mathrm{ad}})_+ \text{ and } \check{\lambda} \geq -w(\mu), \forall w \in \mathbf{W}\right\}.
\end{aligned}$$

It is known that $\overline{\mathbf{T}}_{\mathbf{M}}$ is Cohen–Macaulay: this is a well-known property of affine normal toric varieties; in fact, any normal reductive monoid is Cohen–Macaulay by [Rit03, Corollary 2].

2.3.18. It is also possible to talk about the category $\mathcal{FM}(\mathbf{G})$ for any semisimple group $\mathbf{G}$. In this case, let $\mathbf{G}^{\mathrm{sc}} \to \mathbf{G}$ be the natural isogeny and let $\mathbf{Z}'$ be the kernel; then the universal monoid $\mathrm{Env}(\mathbf{G})$ for $\mathbf{G}$ is none other than the GIT quotient $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}}) /\!\!/ \mathbf{Z}'$, and any $\mathbf{M} \in \mathcal{FM}(\mathbf{G})$ is of the form $\mathbf{M}' /\!\!/ \mathbf{Z}'$ for some $\mathbf{M}' \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$. The details can be found in [Rit01], for example. We may also talk about abelianization map of $\mathbf{M} \in \mathcal{FM}(\mathbf{G})$, and since $\mathbf{M}$ is affine and $\mathbf{G}$ is reductive, it is easy to see that $\mathbf{A}_{\mathbf{M}} = \mathbf{A}_{\mathbf{M}'}$.

We will rarely talk about monoids in $\mathcal{FM}(\mathbf{G})$ for non-simply-connected groups $\mathbf{G}$, for two reasons: the first is that monoids in $\mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ are already sufficient for studying fundamental lemma and many other aspects of representation theory, and the second is that the invariant quotient $\mathbf{G} /\!\!/ \mathbf{G}$ is in general ill behaved (and so is $\mathbf{M} /\!\!/ \mathbf{G}$ by extension) and not suitable for developing a geometric theory. However, one can see §§ 5.3 and 5.4 for a brief use of monoids in $\mathcal{FM}(\mathbf{G}^{\mathrm{ad}})$.

2.4 INVARIANT THEORY OF REDUCTIVE MONOIDS

In this section we discuss the invariant theory of reductive monoids. Most results are built upon ones in § 2.2.

2.4.1. Let $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ be any monoid. We have the adjoint action of $\mathbf{G}^{\mathrm{sc}}$ on $\mathbf{M}$. It can be viewed as the restriction of the $\mathbf{G}^{\mathrm{sc}} \times \mathbf{G}^{\mathrm{sc}}$-action to the diagonal embedding of $\mathbf{G}^{\mathrm{sc}}$. The center $\mathbf{Z}^{\mathrm{sc}}$ acts trivially, so the action factors through $\mathbf{G}^{\mathrm{ad}}$, hence lifts to a $\mathbf{G}$-action on $\mathbf{M}$. The GIT quotient space $\mathbf{M} /\!\!/ \mathbf{G}$ maps to $\mathbf{A}_{\mathbf{M}}$. We also have the *cameral cover* $\pi_{\mathbf{M}}\colon \overline{\mathbf{T}}_{\mathbf{M}} \to \overline{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W}$.

On the other hand, the fundamental representation ρ_i extends to $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ by its definition, still denoted by ρ_{i+}. Therefore, the character function χ_{i+} makes sense for arbitrary $\mathbf{M} \in \mathcal{FM}(\mathbf{G}^{\mathrm{sc}})$ after realizing it as a pullback of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$. Thus, we have a map $\mathbf{M} \to \mathbf{A_M} \times \mathbf{C}$, which factors through the GIT quotient map $\chi_{\mathbf{M}} \colon \mathbf{M} \to \mathbf{C_M} := \mathbf{M} /\!\!/ \mathbf{G}$.

Theorem 2.4.2. *The maps* $\overline{\mathbf{T}}_{\mathbf{M}} \to \mathbf{M} \to \mathbf{A_M} \times \mathbf{C}$ *induce canonical isomorphisms*

$$\overline{\mathbf{T}}_{\mathbf{M}} /\!\!/ \mathbf{W} \simeq \mathbf{C_M} \simeq \mathbf{A_M} \times \mathbf{C}.$$

In fact, the first isomorphism holds for any *(not necessarily very flat) monoid with* $\mathbf{M}^{\mathrm{der}} \cong \mathbf{G}^{\mathrm{sc}}$.

Proof. The first isomorphism is proved in [Ren88], but the reader can also find it in [Ren05] for a more modern reference. We do not know where the second isomorphism was first proved, but it was at least proved in [Bou15]. However, the reader should also see [BC18, Chi22] since [Bou15] contains some error unrelated to the current theorem. ∎

Corollary 2.4.3. *The cameral cover* $\pi_{\mathbf{M}}$ *is a Cohen–Macaulay morphism; in other words, it is flat and has Cohen–Macaulay fibers.*

Proof. Since being a Cohen–Macaulay morphism is stable under base change, it suffices to prove the claim for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, in which case $\mathbf{C_M}$ is regular. Since $\overline{\mathbf{T}}_{\mathbf{M}}$ is Cohen–Macaulay, and $\pi_{\mathbf{M}}$ is finite, it is a flat morphism. Then it is a general result that a flat morphism between locally Noetherian schemes with a Cohen–Macaulay source must be Cohen–Macaulay. See [Sta25, Tag 0C0X]. ∎

Proposition 2.4.4. *Given a Steinberg quasi-section* $\epsilon^{(\xi,\dot{S})}$ *of the group* $\mathbf{G}^{\mathrm{sc}}$*, the map*

$$\begin{aligned} \epsilon_{\mathbf{M}}^{(\xi,\dot{S})} \colon \mathbf{A_M} \times \mathbf{C} &\longrightarrow \mathbf{M} \\ (a,x) &\longmapsto \delta_{\mathbf{M}}(a)\epsilon^{(\xi,\dot{S})}(x) \end{aligned}$$

defines a quasi-section of $\chi_{\mathbf{M}}$ *whose image lies in the regular locus* $\mathbf{M}^{\mathrm{reg}}$*. Moreover, the union of* $\operatorname{Im} \epsilon_{\mathbf{M}}^{(\xi,\dot{S})}$ *over all Coxeter data* $(\xi, \dot{S})$ *generates* $\mathbf{M}^{\mathrm{reg}}$ *under* $\mathbf{G}^{\mathrm{sc}}$*-action.*

Proof. This is a corollary of [Chi22, Lemma 2.2.8, Propositions 2.2.9, 2.2.19] for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, and the general case is deduced by pulling back. ∎

2.4.5. The Weyl group $\mathbf{W}$ acts on $\overline{\mathbf{T}}_{\mathbf{M}}$. Similarly to the group case, a $\mathbf{G}$-orbit in $\mathbf{M}$ is called *regular* if the stabilizer achieves minimal dimension, *semisimple* if it contains an element in $\overline{\mathbf{T}}$, and *regular semisimple* if it is both regular and semisimple.

2.4.6. The regular semisimple locus $\mathbf{M}^{\mathrm{rs}}$ is open and smooth, and can be characterized by a discriminant function extending Disc from the group case. Indeed, we only need to define it for $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$ and pull it back to general $\mathbf{M}$. For $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, let

$$\mathrm{Disc}_+ := e^{(2\rho,0)} \prod_{\alpha\in\Phi} (1 - e^{(0,\alpha)}).$$

This function extends to $\overline{\mathbf{T}}_{\mathbf{M}}$ and is $\mathbf{W}$-Invariant, hence descends to a function on $\mathbf{C}_{\mathbf{M}}$. It is called the *extended discriminant function*, and defines the *extended discriminant divisor* $\mathbf{D}_{\mathbf{M}} \subset \mathbf{C}_{\mathbf{M}}$. The complement of $\mathbf{D}_{\mathbf{M}}$ is the regular semisimple locus $\mathbf{C}_{\mathbf{M}}^{\mathrm{rs}}$, whose preimage under $\chi_{\mathbf{M}}$ is $\mathbf{M}^{\mathrm{rs}}$. For convenience, we will also regard $\mathbf{E}_{\mathbf{M}}$ as a divisor on $\mathbf{C}_{\mathbf{M}}$ by pulling back (and it will not cause any confusion), and denote the *invertible locus* $\mathbf{C}_{\mathbf{M}}^{\times} := \mathbf{A}_{\mathbf{M}}^{\times} \times \mathbf{C}$.

Lemma 2.4.7. *The divisor $\mathbf{D}_{\mathbf{M}}$ intersects $\mathbf{C}_{\mathbf{M}} - \mathbf{C}_{\mathbf{M}}^{\times}$ properly. In other words, the codimension of $\mathbf{D}_{\mathbf{M}}^{\times} := \mathbf{D}_{\mathbf{M}} \cap (\mathbf{C}_{\mathbf{M}} - \mathbf{C}_{\mathbf{M}}^{\times})$ in $\mathbf{C}_{\mathbf{M}}$ is at least 2. In particular, set-theoretically $\mathbf{D}_{\mathbf{M}}$ is the closure of $\mathbf{D}_{\mathbf{M}}^{\times}$.*

Proof. The proof is essentially in [Chi22, Lemma 2.4.2], and we reproduce it here. Without loss of generality, we may assume that $\mathbf{M}$ has 0 (since we can always find a very flat reductive monoid with 0 containing $\mathbf{M}$ by enlarging the cocharacter cone). When $\mathbf{M} = \mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, consider the idempotent $e_{\varnothing,\Delta}$ (see § 2.4.16). It can be explicitly computed that $e_{\varnothing,\Delta}$ is regular semisimple. Its image in $\mathbf{A}_{\mathbf{M}}$ is 0, which is contained in every irreducible component of $\mathbf{A}_{\mathbf{M}} - \mathbf{A}_{\mathbf{M}}^{\times}$. Using the universal property of $\mathrm{Env}(\mathbf{G}^{\mathrm{sc}})$, we see that for any $\mathbf{M}$ the image of the regular semisimple locus is dense in every irreducible component of $\mathbf{A}_{\mathbf{M}} - \mathbf{A}_{\mathbf{M}}^{\times}$, and we are done. ∎

Lemma 2.4.8. *The divisor $\mathbf{D}_{\mathbf{M}}$ is a reduced divisor.*

Proof. Since $\mathbf{D}_{\mathbf{M}}$ is a principal divisor in a Cohen–Macaulay scheme, it is itself Cohen–Macaulay. By Lemma 2.4.7, we only need to prove that it is reduced over the invertible locus $\mathbf{D}_{\mathbf{M}}^{\times}$. But then it suffices to consider the divisor in $\mathbf{T}_{\mathbf{M}}$ cut out by $(1 - e^{(0,\alpha)})(1 - e^{(0,-\alpha)})$ for a single root α, and it reduces to groups with derived subgroup SL_2, in which case we can directly compute. The argument is parallel to, for example, [Ngô10, Lemme 1.10.1]. We leave the details to the reader. ∎

2.4.9. Contrary to the Lie algebra case, there is in general more than one open orbit in the fibers of $\chi_{\mathbf{M}}$; in other words, the fibers of $\chi_{\mathbf{M}}^{\mathrm{reg}} : \mathbf{M}^{\mathrm{reg}} \to \mathbf{C}_{\mathbf{M}}$ are no longer homogeneous $\mathbf{G}$-spaces.

2.4.10. We would like to define the regular centralizer group scheme $\mathbf{J}_{\mathbf{M}} \to \mathbf{C}_{\mathbf{M}}$ similarly to the group case, but the original descent argument needs some adaptation, due to the fact that a fiber of $\chi_{\mathbf{M}}$ may contain multiple regular orbits.

The key observation used by [Chi22] is that the numerical boundary divisor $\mathbf{E}_{\mathbf{M}}$ and the discriminant divisor $\mathbf{D}_{\mathbf{M}}$ intersect properly (Lemma 2.4.7). The descent argument works over $\mathbf{M}^{\times} \cup \mathbf{M}^{\mathrm{rs}}$, which is an open subset whose complement has codimension at least 2. We leave the details to [Chi22, Lemma 2.4.2].

Proposition 2.4.11. *There is a unique smooth commutative group scheme* $\mathbf{J_M} \to \mathbf{C_M}$ *with a* $\mathbf{G}$*-equivariant isomorphism*

$$\chi_{\mathbf{M}}^{*}\mathbf{J_M}|_{\mathbf{M}^{\mathrm{reg}}} \xrightarrow{\sim} \mathbf{I}_{\mathbf{M}}^{\mathrm{reg}},$$

which can be extended to a homomorphism $\chi_{\mathbf{M}}^{*}\mathbf{J_M} \to \mathbf{I_M}$.

There is also a description of the regular centralizer using cameral cover $\pi_{\mathbf{M}} \colon \overline{\mathbf{T}}_{\mathbf{M}} \to \mathbf{C_M}$. As in the group case, let

$$\Pi_{\mathbf{M}} = \pi_{\mathbf{M}*}(\mathbf{T} \times \overline{\mathbf{T}}_{\mathbf{M}}),$$

and

$$\mathbf{J}_{\mathbf{M}}^{1} = \Pi_{\mathbf{M}}^{\mathbf{W}}.$$

Similarly, we have subfunctor $\mathbf{J}'_{\mathbf{M}}$: for a $\mathbf{C_M}$-scheme S, $\mathbf{J}'_{\mathbf{M}}(S)$ consists of points

$$f \colon S \times_{\mathbf{C_M}} \overline{\mathbf{T}}_{\mathbf{M}} \to \mathbf{T}$$

such that for every geometric point $x \in S \times_{\mathbf{C_M}} \overline{\mathbf{T}}_{\mathbf{M}}$, if $s_\alpha(x) = x$ for a root α, then $\alpha(f(x)) \neq -1$. With the same proof as in the group case, this is an open subgroup scheme of $\mathbf{J}_{\mathbf{M}}^{1}$ containing the fiberwise neutral component $\mathbf{J}_{\mathbf{M}}^{0}$.

Proposition 2.4.12. *There is a canonical open embedding*

$$\mathbf{J_M} \longrightarrow \mathbf{J}_{\mathbf{M}}^{1}$$

that identifies $\mathbf{J_M}$ *with* $\mathbf{J}'_{\mathbf{M}}$.

Proof. The open embedding claim is proved in [Chi22, Proposition 2.4.7]. We even have $\mathbf{J_M} = \mathbf{J}_{\mathbf{M}}^{1} = \mathbf{J}'_{\mathbf{M}}$ if $\mathbf{G} = \mathbf{G}^{\mathrm{sc}}$. The point is that the complement of $\mathbf{C}_{\mathbf{M}}^{\times} \cup \mathbf{C}_{\mathbf{M}}^{\mathrm{rs}}$ has codimension 2, thus we only need to prove the claim over this open locus; but then it is a consequence of the group case and the regular semisimple case, both of which are easy. ∎

Remark 2.4.13. In [Chi22, Proposition 2.4.7], the open embedding $\mathbf{J_M} \to \mathbf{J}_{\mathbf{M}}^{1}$ relies on the Grothendieck's simultaneous resolution for monoids, which is analogous to the well-known group case:

$$\begin{array}{ccc} \widetilde{\mathbf{M}} := \mathbf{G}^{\mathrm{sc}} \times^{\mathbf{B}^{\mathrm{sc}}} \overline{\mathbf{B}}_{\mathbf{M}} & \xrightarrow{\tilde{\pi}_{\mathbf{M}}} & \mathbf{M} \\ \downarrow & & \downarrow \\ \overline{\mathbf{T}}_{\mathbf{M}} & \xrightarrow{\pi_{\mathbf{M}}} & \mathbf{C_M}, \end{array}$$

where $\overline{\mathbf{B}}_{\mathbf{M}}$ is the closure of the Borel $\mathbf{B_M}$ of $\mathbf{M}$ induced by $\mathbf{B}$. The crucial part is to define the vertical map on the left by defining a canonical projection map $\overline{\mathbf{B}}_{\mathbf{M}} \to \overline{\mathbf{T}}_{\mathbf{M}}$. However, the definition of such projection in [Chi22] is wrong. The correct definition

(continued...)

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