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PRELIMINARIES

CHAPTER ONE

Beginnings

THE EARLY DAYS

Many findings in archaeology bear witness to some math in the mind of our ancestors. There are many scholarly books on that matter, but we may be content with a few examples. A bone rod, which was discovered in 1937 in Moravia, shows 55 notches in groups of five and is about 30,000 years old. Paintings on cave walls and many engraved objects show various forms of geometric design, more and more sophisticated as one approaches the beginning of the Neolithic period (12,000 years ago, when agriculture began).

Mesopotamian, Egyptian, Indian, Chinese, and Old American civilizations already knew as much math as is taught in the first grades of our high schools. Basic arithmetic, including the four operations, is carefully described in Babylonian tablets and Egyptian papyri. The many exercises accompanying the descriptions show that mathematics was then an empirical science: the examples are very practical, showing, for instance, how one can divide a herd into so many equal parts (quite useful in a case of inheritance or when sharing some plunder). Prime numbers then begin to appear. Mesopotamian, Egyptian, and Chinese scribes knew much geometry. Figures 1.1–1.3 give examples of their knowledge. Figure 1.1 shows how the Egyptians computed the area of a

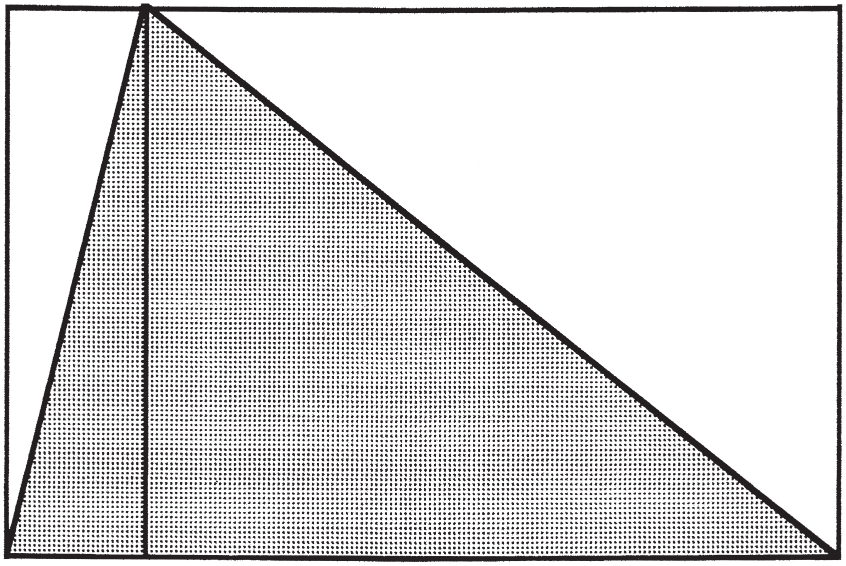
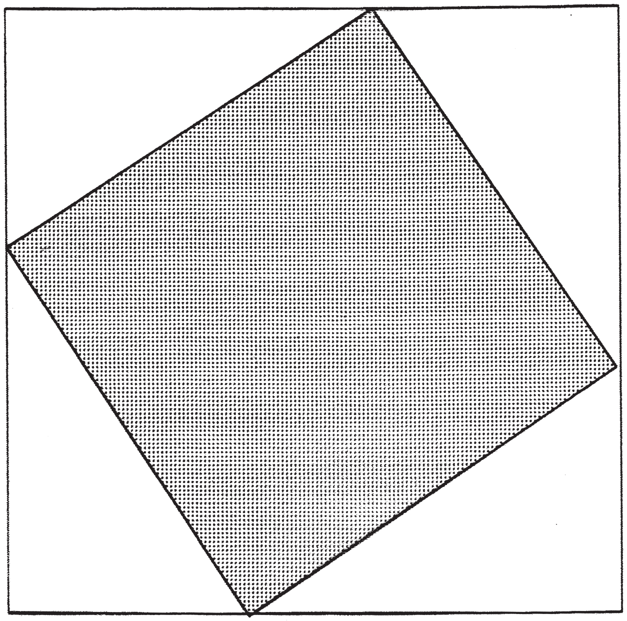
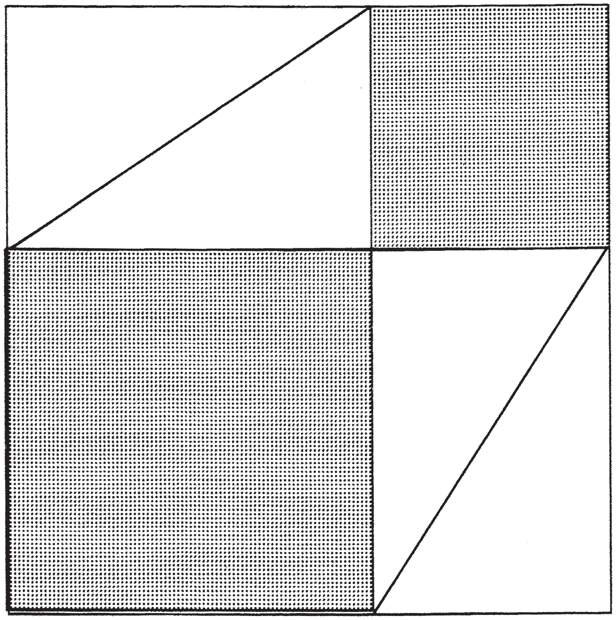


Figure 1.1. How the early Egyptians computed a triangle area. This drawing shows that the triangle area is half the area of the rectangle built on its base and height.

triangle. Herodotus, the Greek historian, explains that Pharaoh's administration taxed land on the basis of acreage, and one sees how practical this kind of recipe could be. Babylonians knew the so-called Pythagorean theorem (involving the sides of a rectangular triangle), probably on the ground of some drawing like figure 1.2 (figure 1.3, which is similar, is from a later Chinese source). One could also mention good approximations of π, the volume of a pyramid, the solution of second-degree algebraic equations, and a few other items, but they would not add much to the basic statement: much mathematics was known early and always for practical reasons with empirical means.

Figure 1.2. A drawing yielding the Pythagorean theorem. By moving triangles, one concludes that the shaded square area in (a) is the sum of the two square shaded areas in (b).





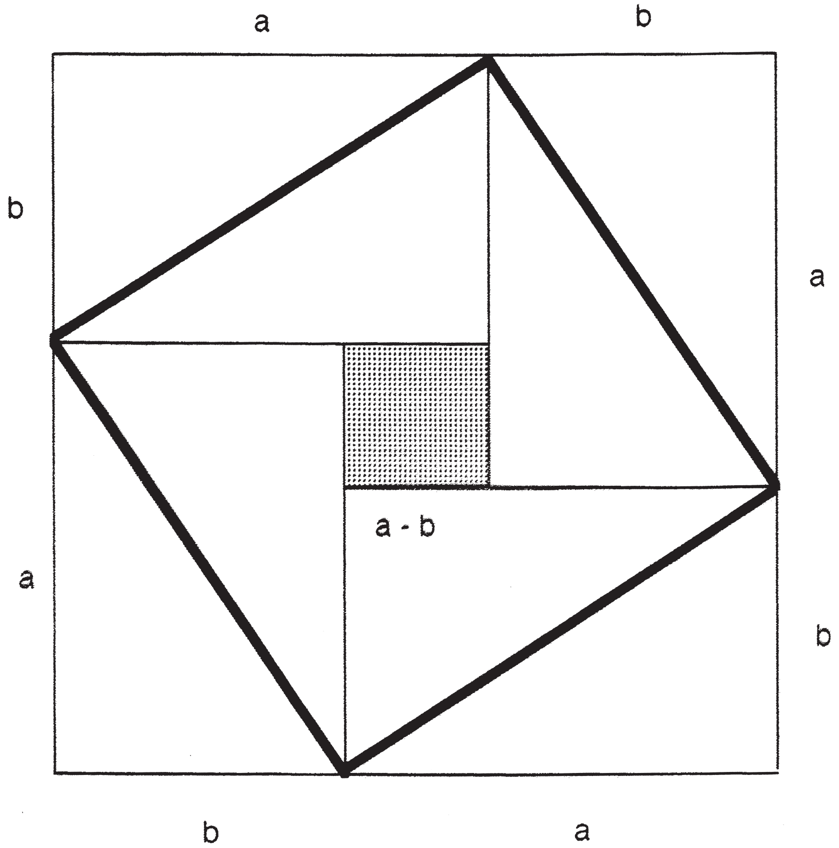


Figure 1.3. A Chinese version of the Pythagorean theorem (around 200 B.C.). If a and b denote the sides of a right triangle, the boundary square has a side $(a + b)$ and the smallest shaded one a side $(a - b)$. The side c of the blank triangle is obviously such that $c^2 = (a - b)^2 + 4(ab/2) = a^2 + b^2$.

THE BEGINNINGS OF GREEK MATHEMATICS

Babylonian and Egyptian texts show that the early mathematicians were working on bookkeeping, agronomy, architecture, or astronomy, but one does not know what dreams they entertained around their art. Greek mathematics begins, on the contrary, with the flamboyant figure of Pythagoras (c. 580–500 B.C.).

Had he lived one century earlier he would have come to us as a legendary personage, like Orpheus or Theseus. Myth had already begun haloing him, since he was said to have a thigh made of gold. He founded a quasireligious sect that still existed in Plato's time, 150 years later. Little is known of his doctrine, but he certainly held that "numbers govern everything," whatever that means; he was eagerly interested in mathematics and he is said to have sacrificed an ox when he discovered (or perhaps proved by new means) the Pythagorean theorem on rectangular triangles.

The earliest mathematicians were certainly Pythagoreans and an interesting speculation is mentioned in Bourbaki's *Elements of a History of Mathematics*, which would make math begin like a novel (von Fritz 1945). It may be entertaining to assume this story is true, as follows: One of the main symbols of the Pythagoreans was the pentagram, the stellar regular pentagon that was always treasured by all sorts of mystical groups. They knew how to construct it inside a circle with a compass and a ruler, and the construction shows easily that the ratio of the stellar pentagon side to the radius of the circle is the "golden ratio" $a = (1/2)(\sqrt{5} + 1)$. This ratio played a great role in Greek aesthetics, including painting, sculpture, and architecture, and it was certainly one of the numbers governing the world, according to the Pythagorean doctrine. This doctrine held, however, that the numbers worth considering are integers or made of integers. The golden ratio should therefore have been a quotient $a = p/q$ of two integers p and q. One could divide the two of them by a common divisor until neither was left. And then comes the drama! The obvious relation $a = 1/(a + 1)$ implies that $p(p + q) = q^2$. But that means that p and q must have a common divisor, which is impossible. Something was then wrong in Pythagoras' mystical assertion.

Of course, this story is invented or at best speculative, but one knows for sure that some Pythagoreans investigated the diagonal of a square and proved without any possibility of escape that $\sqrt{2}$ cannot be rational (i.e., a quotient of integers).

That was a hard blow for Pythagoras' doctrine. It occurred probably early in the fifth century B.C., but one does not know who found it. Did one man discover it, or maybe a woman, since the sect accepted women? Or was it the outcome of long investigations by a group of people? We don't know. We know that a man, Hippasus of Metapontium, was accursed for letting the secret of the result leak out, but that does not mean that he discovered it.

On the other hand, one cannot underestimate the importance of the event. It was certainly the first true theorem, the first one at least that could not be made obvious by means of a clever drawing. It implied to Greek eyes that the mind is able to reach a hidden truth by itself, using only the power of its own thought. It revealed the power of logic, which was still in the turmoil of initial searches. It increased the Greek confidence in the supremacy of ideas and, incidentally, it also led for a long time to disparagement of the empirical approach to science. Last but not least, it showed the value of rigor, which led the way toward Euclid's axiomatic construction of mathematics.

Plato and the Philosophy of Mathematics

There is no indication of a specific philosophy of mathematics after the failure of Pythagoras' attempt. The main questions were concerned with general philosophy: what is Being, and should there be non-Being, is there infinity, what are Good and Beauty, what is reason (although people preferred speaking of the nature of ideas), what is life, and many other such issues. Mathematics entered that game with Plato in a rather roundabout way, more as an example than for its own sake, but it was the beginning of a long story.

Plato (c. 428–348 B.C.) knew well the mathematics and the mathematicians of his time, including particularly Eudox, who was his student and friend. One of his main interests was, however, the meaning of ideas. He often took mathematics as a paradigm and one of his early dialogues, the *Meno*, gives a good indication of his early views on that matter. At some point in the dialogue, Socrates asks a boy various mathematical questions such as: How much larger is the area of a square when the side is multiplied by two? How big is the square built on the diagonal? The boy is supposed to know nothing, since he was born a slave. . . . Socrates shows himself, however, a very kind inquirer, he gives many clues, he suggests which lines can be drawn to get a hint, and every student would certainly get an A grade with such an examiner. The boy answers correctly, of course: he is led so gently, and a modern reader would simply conclude that he does not lack common sense. That is not Plato's conclusion, however. The answers prove that the boy knew them before he was asked the questions, and the query only helped him to remember them!

This example was typical of Socrates' method, which Plato learned in his youth. The Pythagorean School influenced him later, and mathematics then became a more important key in his philosophy. His main assumption was the existence of two different worlds. There is the world we see with our eyes and in which our body is immersed, a world that can be considered as more or less trivial according to Plato. There is also another world in which perception is replaced by understanding, and mathematics originates from it. Our senses cannot reach that world, which is supposed to be more real than the one we live in, and its inhabitants are immaterial. They are the Forms, or Ideas. They belong to the sphere of divinity, higher than the gods themselves, and they share a common harmony including everything that is "Good."

Plato considered, for instance, that the world of Ideas contains a Form "circle" embodying all the possible circles in the world below, and similarly a Form "triangle." He was much impressed by the fact that mathematical properties can be discovered though nothing hints at them in the definitions, the fact, for instance, that the median lines of a triangle meet at a common point at one-third of their length. That property was already there before any worldly triangle had been drawn for the first time. This feeling of a preexistence, of something more real than reality, has always impressed many mathematicians, and one may presume that Eudox, who was one of the greatest mathematicians of all times, confirmed Plato on that point. There are still many believers or would-be believers in this form of Platonism among modern mathematicians, who feel something like the existence of another world where mathematical truth rests.

Plato was aware of an obvious objection to his proposal: How can we, we people made of flesh, who live in this world down here, how can we get in touch with the ideal world where mathematical truth dwells? His answer was that our soul inhabited that world before we were born, and we have memories of it. We may of course forget this answer, but the question itself will remain interesting.

Aristotle and Abstraction

Aristotle (c. 385–322 B.C.) brought mathematics back down to earth. He considered the mathematical objects, numbers, circles, triangles, and so on, as so many abstractions of real objects, either natural or manmade. Although every line we draw with a stiletto on a waxed plate has a finite breadth and irregularities, our mind can make an abstraction of them, forget them, and consider them as irrelevant. Irrelevance is the motto when somebody worries about giving the same name to two obviously different triangles: everything making them different is inessential. Aristotle in that sense considered the mathematical objects as very close to natural objects, or at least as patterns that are found in reality.

Plato's problem, "Why are there mathematical properties that are not contained in the definitions?" received a new answer: Logic can create new truth, and this kind of property gives a perfect example of its ability. One should not forget that the discovery of logic was still recent, and Aristotle was one of its major investigators. Some of his concepts are worth mentioning and, rather than choosing them in his *Logic* or his *Metaphysics* to which I intend to return, I will pick them up in his *Physics*. He says in that book that we cannot really understand something without knowing its first principles. He enters then into various predicaments about Being and non-Being, about motion as a transition from being there to not being there. He states as a principle that every motion must proceed from a permanently active cause (a principle that, by the way, impeded physics for a millennium and a half), so that a moving object is moved by another, which is also moved, and so on, until one must arrive finally at a primary mover, who pushes the sphere on which the stars are nailed in a perfect motion that is necessarily rotation. Aristotle's book is a work of beauty and also an ascetic song of love for nature, *physis*, since love and hate are among its other basic principles: a stone falls because of its love for the earth and smoke rises up for love of the sky.

Philosophers enjoy that book for the tension in its argument and they do not worry that most of its conclusions have turned out to be wrong. Physicists would rather say that it is not a physics book in spite of its title. I wished to mention it however, because of its relation to the main topic of the present book and particularly in view of two significant statements by Aristotle, namely, (i) mathematics relies primarily on an abstraction of reality; (ii) physical reality can be understood only by getting at its first principles. These statements could look like our thesis of physism in a nutshell, except that mathematics, the principles of physics, and even the meaning of reality have much changed in the meantime.

CHAPTER TWO

Brain and Reality

ABSTRACTION AND THE BRAIN

The essence of mathematics appears to be a study of the *relations* between some objects, which are (voluntarily) known and described through only *a few* of their properties" (Bourbaki 1960). This strong statement implies that abstraction is the keyword in math, a dominance that is made still clearer when Bourbaki adds that the "few properties" to be retained in the study of mathematical relations are "the axioms at the basis of the theory." Abstraction is therefore supposed to dig deeper and deeper and to strip mathematical objects of more and more of their properties, until some ultimate axioms are reached.

The purpose of the present chapter will therefore be to study abstraction. It will not be considered abstractly however, but plainly, as something our brain does more or less casually. Among the many trends in the philosophy of mathematics, we begin therefore with one of the latest: the cognitive approach, which reminds us essentially that math is made by the mathematician's brain.[1] "Mathematics as we know it is . . . a product of the human mind. . . . It comes from us! We create

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[1] See, for instance, Changeux's contribution in Changeux and Connes (1995).

it, but it is not arbitrary [because] it uses the basic conceptual mechanisms of the embodied human mind as it evolved in the real world. Mathematics is a product of the neural capacities of our brains, the nature of our bodies, our evolution, our environment, and our long social and cultural history."[2] These statements would have looked trivial and useless a few decades ago, but the present advances in brain science begin to make them helpful: To know, for instance, that abstracting is one of the most commonplace performances of the brain is certainly something new and significant. Psychologists would have considered it earlier as a highly evolved operation, and that makes things very different.

The present chapter will draw much, therefore, from discoveries in neural science. It is strongly influenced by the cognitive approach to mathematics, but it will also draw our attention to the obvious fact that the brain is something real, perceiving real things, so that reality is prior to cognition. Well, doesn't all that sound trivial enough? It's time to show that there is more in it than meets the eye.

THE QUESTION OF REASON

The ancestors of the cognition scientists were John Locke and David Hume, and they were not much interested in mathematics. Their empiricism, however, which wants to explain the origin of ideas and language as well as the meaning of reason, will often provide us with a useful background. Ideas, for instance, according to Hume, "are copies of our [most lively perceptions.] . . . Every idea is copied from some preceding impression or sentiment." Hume also noticed that "though our thought seems to possess [an] unbounded liberty, we . . . find, upon a nearer examination, that it is really confined within

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[2] Lakoff and Nuñez 2000, 9. (This quotation is borrowed from Henderson 2002.)

very narrow limits and that all this creative power of the mind amounts to no more than the faculty of compounding, transposing, augmenting, or diminishing the materials afforded us by the senses and experience." Concerning language, one may quote Locke: "Words become general by being made the signs of general ideas; and ideas become general by separating from them the circumstances of time and place, and any other idea that may determine them to this or that particular existence. By this way of abstraction they are made capable of representing more individuals than one."

Roughly speaking, empiricism considers that reason results from an adaptation of human beings to the universal regularities in nature. Language is also a by-product of these regularities, a wonderfully clever way of expressing the information from them. Reason and language proceed through an abstraction of differences among individual cases and a selection of their common patterns. Some other aspects of early empiricism have become partly obsolete—its reduction of physical laws to a constant habit, for instance. But we had better leave out general philosophy at this point.

The cognitive sciences have much refined empiricism. They take advantage of many contributions by philosophers, linguists, mathematicians, computer scientists, physicists, and of course, prominently, biologists and physicians. An essential early improvement was to take evolution into account, so that millions of years since the origin of humankind became available for explaining the development of language and reason, rather than the short span of a human life. Evolution grants, moreover, hundreds of millions of years, since every animal species, including bees, whales, and chimpanzees, has a form of language using sounds, gestures, or smell (this line of research is also presently very active). Although one cannot speak properly of reason, every animal is able to gather information from the regularities of its surroundings and utilize it (the name "Information Gathering and Utilizing System," IGUS for short, has been coined for this by Murray Gell-Mann). The scope of the idea is even wider since it extends to plants and bacteria, thus allowing billions of years for the slow rise of reason. Our search for the meaning of mathematics is of course narrower, but it can ignore neither brain science nor the framework of nature, from which every kind of information is gathered.

Perception

The brain gains information from reality through perception. Brain science by now provides an impressive amount of data on perception and its mechanisms, but it will be enough to mention a few significant results concerning vision, which is the most convenient example. One knows from physics how an object emits light or reflects it, and the laws of optics can also explain how light produces an image on the retina after crossing the optical part of the eye. Then perception really begins. The retina involves 140 million photoreceptors, which are surrounded by an interconnected net of neurons. These neurons are essentially an outside part of the brain; they send their visual information to the cortex through the optical nerve, which consists of a million or so long fibers (or axons) belonging to a family of transmitting neurons. Each axon carries electrical pulses, and the collection of these signals provides a message, which is sent to the cortex and provides it with all the information that the brain will have about the image. One might say that this message is coded, since the initial optical data have been transformed into electric data, somewhat analogous to the electromagnetic signals of a television network.

It was believed for a long time that the retinal neurons transmit the image directly to the brain and preserve its shape and color exactly, but they actually perform a much more elaborate work of filtering and coding. As a matter of fact, the information inside the retina is too large to be transmitted usefully: We saw how huge the number of photoreceptors is and there are about as many neurons, each neuron emitting and receiving pulses. This accumulation of little signals would produce a mess if it arrived directly at the brain and, furthermore, it could not be carried through the few fibers (a mere million) in the optical nerve. Some order must be introduced and it comes from synchronism: It has been found that many neurons can cooperate when they adjust their pulses to an exact simultaneity, while the random pulses originating from noncooperating neurons have no significant effect. The information reaching the brain is only the outcome of large bunches of synchronous signals originating from cooperating families of neurons.

The synchronous action of neurons has an immediate consequence on our representation of reality. Without knowing how the brain constructs this representation, one may be sure that it must be governed by time, ordered in time, and practically instantaneous, since the information generating it has this character. This biological priority is obviously reminiscent of Kant's vision of time as a "pure form of intuition": something intrinsic to reason. He went too far, however, when he excluded time from reality and regarded it "as a condition of the possibility of phenomena, not as a determination produced by them." It may be important to recognize, anyway, that we possess an embodied *intuition* of reality, which does not necessarily reflect the true features of reality and remains a result of the work of the brain.

The collective mechanism of neurons also has a consequence on our intuition of space. It has been observed that various groups of neurons working together are associated with small circular regions of the retina. As a consequence, the retinal image is spread over these regions when the information arrives at the brain. Perception therefore enforces a *continuous representation* of reality, which is not intrinsic to the retina itself, since each photoreceptor can detect light photon by photon. This detailed information is thrown out, however, when many neurons adjust together to emit synchronous signals. It may be added that, when we recall the memory of an image, the zones at work in the cortex are the same as when the image is actually seen. Our imagination is therefore biologically constrained to continuity, and this simple observation will enlighten a few significant points in the history of mathematics when we come to that.

The signal incoming from the retina is analyzed in more than thirty regions of the brain, each one having a definite localization and a specific function. Some of them recognize straight lines, and different zones react to horizontal or to vertical lines. Other regions are specialized in the analysis of color, or in detecting motion, or various other aspects of images. The different regions also exchange their information during the process, and there exists a complex of regions where the recognition of objects takes place. The theory in best agreement with experimental findings assumes that this pattern recognition involves a comparison by the brain of what is actually seen with what has been previously memorized. Memory is not perfect, however, and the brain remembers only, for instance, some features of a tree it has seen and not every detail. If one puts together this imperfection of memory with the decomposition of an image into various component by the cortex, one may fairly presume that abstraction is one of the most fundamental and ordinary process of the brain. This is certainly interesting when one turns to mathematics, and also a good point in favor of Aristotle against Plato.

What about consciousness? It is poorly understood but some experiments show a few interesting features in it. One may project, for instance, two different images that are seen by the two eyes of a cat: the left eye sees an unmoving bee and the right one a moving butterfly. A mirror standing right on the nose of the animal makes its brain see two superposed images, because of binocular vision. Electrodes in the cat's brain indicate what he is conscious of seeing (or at least where his attention is caught): if the left eye is seeing consciously for instance, the neuronal signals emitted by the cortex regions that are associated with that eye become synchronized, whereas the signals originating from the right eye do not. In these conditions, the brain never shows a simultaneous awareness of the two eyes, and only one of them is responsive. This is to be contrasted with the simultaneity of other unconscious perceptions, indicating for instance that the bee is perceived though not seen when the cat's attention is concentrated on the butterfly. This observation is very important, since it indicates that consciousness—or awakening, or whatever it is—implements uniqueness in the brain's representation of reality.

A last datum from neurology is also relevant for our topic. It is concerned with pattern recognition and, once again, I will give only one example. Figure 2.1 shows a drawing that

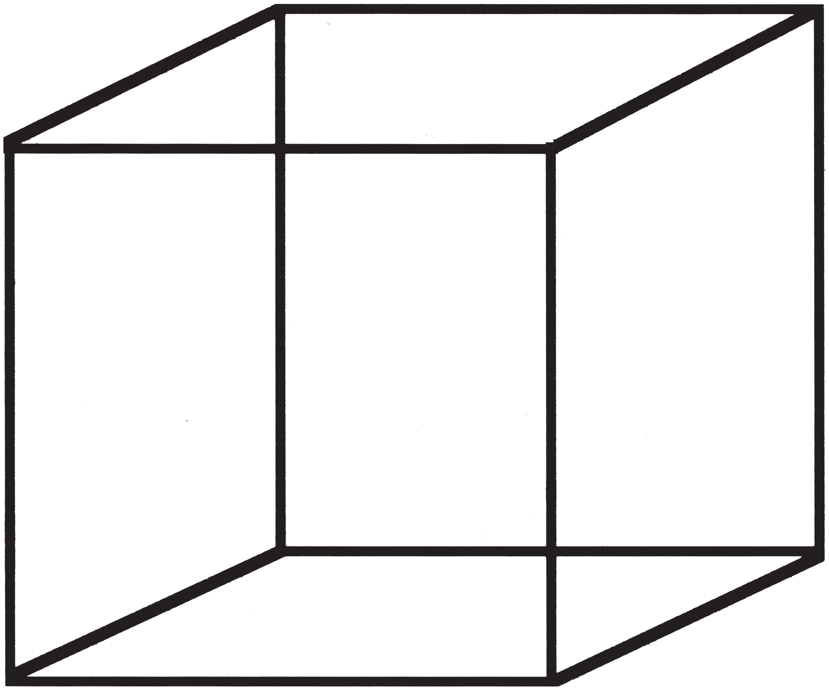

Figure 2.1. An optical illusion. We see the drawing as representing a cube, but cannot tell which face is forward or backward.

our brain recognizes spontaneously as the image of a cube. Because of a lack of perspective, we cannot tell, however, which face of the cube is forward and which is backward. A remarkable physiological fact is that our consciousness always decides which is which, but the decision can switch from one choice to the other: one face of the cube seems to stand in front at some time and a moment later, suddenly, the opposite face pops up in its place. Many similar experiments have been made under various conditions on humans and monkeys, including some with the help of magnetic resonance imagery, and they agree on a common conclusion: Pattern recognition is a universal ability in higher species, and the brain systematically selects a unique pattern at a given instant.

Our brain always separates a scene into definite objects and rejects ambiguities, with no intervention of the will. Sometimes there is interplay of ambiguities and spontaneous decisions, as when we hesitate about the exact nature of an object far away or badly illuminated. The phenomenon fascinated some Italian painters during the Renaissance period and they produced clever drawings and paintings compelling the spectator's consciousness to see different patterns alternately: a man's face or a garden, a bunch of leaves in a tree or a cat waiting, or many other illusions. Salvador Dali, in recent times, pushed the game very far in some of his paintings. There are even cases when the ambiguities of a geometric drawing are so compelling that our brain gets the impression that something in it is moving. The Hungarian painter Victor Vasarely has realized startling paintings on this principle.

THE CATEGORIES OF REALITY AND CLASSICAL SCIENCE

Brain science shows that our representation of reality is strongly constrained by our anatomy and the connections of our cells. If we avail ourselves of an analogy with computers, we might say that the hardware of our brain involves 100 milliard neurons, each neuron having typically ten thousand connections with others. This hardware constrains the software of the brain and its conscious output. Our intrinsic or "biological" representation of reality is thus endowed with several important properties: uniqueness (reality is unique at every moment); localization in time with definite present and past; location also in ordinary ("Euclidean") space (as shown by experimental studies of spatial representation that were not mentioned here). This intuitive representation of reality is continuous and separated into individual objects (never a bee and a butterfly in the same place!). Other investigations, involving actions and not only perception, by humans or higher animals, show an expectative behavior in them, which amounts to an imprint of causality in the brain. One might add also a distinction of reality from the effect of imagination or a virtual appearance, which is manifested in the different reactions of glands and muscles.

Uniqueness, location in ordinary space and time, continuity, separation of phenomena, causality, and a sense of reality: a philosopher would immediately recognize in this list several "categories of reason," as Kant spelled them out. They are constraints of our intuition and not necessarily of reality, as we shall have occasion to discuss. They are not constraints on the abilities of reason, and mathematics, for instance, makes fruitful use of noncontinuous objects. Relativistic physics considers an indivisible space-time, rather than a distinct pair consisting of ordinary time and commonplace Euclidean space. Quantum mechanics, when dealing with atoms and particles, gives up practically every intuitive character of reality. We may thus anticipate that the chasm between our inborn intuitive representation of reality and the concepts of modern science is important for understanding the high abstraction of modern physics and mathematics. It will be convenient in this perspective to introduce a definition of "classical science" that will be used systematically in this book:

Definition. A science is called "classical" when its concepts agree with the characteristics of the intuitive representation of reality, namely, uniqueness, location in ordinary space and time, continuity, separation of phenomena, causality, and a clear-cut distinction between the real and the virtual.

CHAPTER THREE

Mathematics and Classical Reality

The subject of this book being the relation between mathematics and reality, the present chapter will be devoted to a few landmarks in the history of this relation. The questions to be considered will be in point of fact: What has been the relation between mathematics and the *classical* conception of reality, and how has it evolved? It has evolved much, actually, and that will lead us to recall a few landmarks in the history of mathematics and also a few points about the evolution of physics (I believe, as a matter of fact, that a remarkable discordance between the courses of these two sciences during the last hundred years or so has much to do with the present confusion in the philosophy of mathematics).

The Elements of Logic

We begin in the third century B.C. with logic, which is inseparable from mathematics. Experts apparently agree that logic began as an abstraction from ordinary language (see, for instance, Kneale and Kneale 1978). Efficient patterns of thought were recognized inside language, so compelling that they seemed able to reach every possible truth. This is at least how logic arose in Greece and how it linked with mathematics in the form we know it. Logic's name means the science of *logos*, which itself initially meant language but was soon endowed with grand ontological connotations.

We saw in the previous chapter that abstraction is a standard process of the human brain, and it was easy for the early thinkers to abstract some patterns of argument from the structures of language. Rhetoric, which is its primary version, had come earlier, and it had been taught for a century or so—for the sake of winning an audience's favor—when logic came out. Aristotle's idea of considering a proposition as an undivided object and denoting it by a symbol (a letter) was an essential discovery, as well as introducing variables in a proposition such as "*x* is a man."

Negation, which associates a proposition with its opposite (such as "*x* is not a man" in our example) warrants a brief comment. It was first associated in Greek philosophy with the opposition of being and nonbeing, as some lessons of Aristotle remind us. In his discussion of motion, for instance, an object is presently at some place and therefore not in another place: the one it occupied a short while ago. Motion is thus defined as a transition from being (there) to nonbeing, or the other way round. This approach looks strange at first sight, but it becomes perfectly sound when considered from a cognitive standpoint. An important ability of the mind is indeed to compare a situation that is perceived with another that has been kept in memory, and many experiments have studied this process. The rejection of a proposition as incompatible with another is therefore embodied in our brain, and one of its earliest occurrences during evolution was probably the difference between the prey being there and the memory that it was not there a moment before.

Then there is the principle of noncontradiction, which is acknowledged as Aristotle's most important contribution to logic. It says that two propositions, *a* and non-*a*, cannot be true simultaneously. It seems also to be imprinted in our mind, but consciousness is essential to it. We saw in the previous chapter the case of a cat that looks at a bee with the left eye and at a butterfly with the right eye. The motion of his eyes and the synchronization of his neurons indicate that, biologically, he can direct his attention only to a unique insect at a given instant. Similar experiments on human beings show that their consciousness requires the same uniqueness of purpose. Our conscious memory has the same kind of constraint, and one may thus assume that the principle of noncontradiction states a property of our brain. One may even go one step further. Some ambiguous situations, such as a cat being forced to see a bee and a butterfly in the same place or a person contemplating a tricky painting by Salvador Dali, are only a consequence of some artifacts: the tricks of an experimenter or the art of a playful painter. This kind of ambiguity never occurs in macroscopic physics, which is the objective origin of our representation of reality, and the fact that empirical reality is unique is therefore the true foundation of the principle of noncontradiction.[1]

Aristotle's elaboration of logic rested on syllogisms, which relied essentially on the existence of categories (x is a man: it belongs to the category of men) or of properties (x is red: it has the property of exhibiting a red color). Categories and properties are obviously related to nature's regularities and, ultimately, they result from the existence of natural laws. Our first conclusion is then pretty evident: the first elements of logic are direct consequences of the laws of nature through the channel of our perception and because of the constitution of our brain, which developed as an information gathering and utilizing system under the government of the laws.

Greek Mathematics and Logic

Greek mathematics relied on much fewer concepts than did philosophy. Its universe of discourse—as logicians would say—was well defined and it generated a specific form of logic,

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[1] It will be seen later why this statement is far from trivial.

somewhat different from Aristotle's system of syllogisms. Euclid's *Elements* bears witness to this logical autonomy of mathematics, but it is only a matter of form. The remarks we made about the inborn origin of Aristotle's concepts obviously remain valid in the case of Greek mathematical logic.

The foundation of Euclid's mathematics on definitions, axioms, and postulates was probably the result of a systematic exertion by mathematicians, who were often also philosophers and teachers. As philosophers, most of them had an inclination for Platonism or a weak form of it, in which mathematics reflected some kind of divine thought. As teachers, they felt secure behind a sure line of defense in front of their paying students, who were fond of asking incisive questions. After two centuries or so, the outcome was the masterpiece of Euclid's books.

The three species of basic assumptions are clearly inspired by reality and our mental representation of it, which relies on the brain's capacity for abstraction. The main definitions of geometry, such as points, straight lines, planes, and circles, are abstracted from nature and art. Conical curves, which are defined as intersections of a cone of revolution with a plane, are tangible. Real numbers were trickier and they were finally defined through an analogy with the points on a straight line. Algebra, which appeared only in the second century A.C., was clearly an abstraction of arithmetical procedures. It was a kind of arithmetic in which the numbers occurring in an operation were abstracted or left free for choice.

Axioms were supposed to be some obvious propositions, such as, for instance, $a + b = b + a$. Evidence, or obviousness, is most often an inability or a reluctance of our minds to think otherwise: when it is not a result of education or belief, it is very close to biological intuition. Postulates were propositions that were found necessary for building mathematics, but not quite obvious. The most famous examples are the postulates regarding parallel lines, and the corresponding misgivings clearly suggest second thoughts about reality itself: how could one reconcile the infinity of straight lines with the concept of a closed universe? Infinity (*apeiron*) had been introduced before Socrates' time by Anaximander, and Zeno of Elea pointed out its apparent paradoxes. His story of light-footed Achilles running after a tortoise is famous: Achilles always needs some time for crossing half the distance separating him from the tortoise at every moment, which means that the race can be split into an infinite number of time intervals. Zeno wrongly associated this infinite sum with an infinite time, whereas the geometric series $1/2 + 1/4 + 1/8 + \cdots$ for the time to cross half, then half the half, and so on, of the distance to the tortoise converges to 1. Infinity was considered mysterious and a matter rather for philosophy than mathematics. Euclid invoked it only in the "method of exhaustion," defining the number π by means of regular polygons tending toward a circle. Archimedes was bolder and performed real integrations involving a sum of infinitesimal quantities, but his example remained practically unique for a long time.

THE GOLDEN AGE OF CLASSICAL MATHEMATICS

If a "paradise" were ever intended for the enjoyment of mathematicians (and physicists), it was certainly inhabited during the seventeenth and eighteenth centuries, the time of the Enlightenment, when everything in science was finding harmony: geometry with algebra, algebra with calculus, and calculus with dynamics. The considerations of knowledgeable historians of mathematics (e.g., Bourbaki 1960; Dieudonné 1978) can be summarized by defining this period as *physicalist* and *classical* (when using the word "physicalist" in this circumstance, I do not mean only that physical reality and the problems of physics were the main source of inspiration for mathematics, but also a general attitude toward foundations that will become physism in this book). Both algebra and analysis yielded many remarkable results but they could not receive a safe axiomatic

(continued...)

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